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A new formulation of Harnack's inequality for nonlocal operators

Une nouvelle formulation de l'inégalité de Harnack pour des opérateurs non-locaux

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ABSTRACT

We provide a new formulation of Harnack's inequality for nonlocal operators. In contrast to previous versions we do not assume harmonic functions to have a sign. The version of Harnack's inequality given here generalizes Harnack's classical result from 1887 to nonlocal situations. As a consequence we derive Hölder regularity estimates by an extension of Moser's method. The inequality that we propose is equivalent to Harnack's original formulation but seems to be new even for the Laplace operator.

RÉSUMÉ

Nous fournissons une nouvelle formulation de l'inégalité de Harnack pour des opérateurs non-locaux. En contraste avec les versions précédentes, nous n'avons pas à supposer que les fonctions harmoniques sont de signe constant. La version de l'inégalité de Harnack donnée ici généralise le résultat classique de Harnack datant de 1887 pour les cas non-locaux. Conséquemment, on obtient des estimations de la régularité Hölder grâce à une extension de la méthode de Moser. L'inégalité que nous proposons est équivalente à la formulation originale de l'inégalité de Harnack mais semble être nouvelle y compris pour l'opérateur de Laplace.

1. Introduction

Harnack's inequality from 1887 for harmonic functions can be formulated as follows:

Theorem 1.1. *(See [6].) There is a constant $c > 0$ such that for any nonnegative harmonic function $u : B_1 \to \mathbb{R}$*

$$u(x) \leqslant cu(y) \quad \text{for } x, y \in B_{1/2}.$$

In the first fifty years after publication the inequality itself did not attract as much attention as the resulting convergence theorems. The situation nowadays is quite different for at least two reasons. On one hand, the parabolic version of Harnack's inequality guarantees that the underlying manifold satisfies the volume doubling property and a scale invariant Poincaré

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inequality. On the other hand, Moser showed in 1961 that Harnack's inequality itself leads to *a priori* estimates in Hölder spaces. This result can be formulated in a metric measure space (Ω, d, μ). For $R > r > 0$, $x_0 \in \Omega$, set

$$B_r(x_0) = \{x \in \Omega \mid \mathrm{d}(x_0, x) < r\}. \tag{1}$$

For every $r > 0$, let $\mathcal{S}_r$ denote a family of bounded Borel measurable functions on Ω satisfying $\mathcal{S}_s \subset \mathcal{S}_r$ whenever $0 < r < s < \infty$, and $\lambda u + \mu \in \mathcal{S}_r$ for all $r > 0$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$ and $u \in \mathcal{S}_r$.

Theorem 1.2. *(See [10].) Let $x_0 \in \Omega$, $\theta > 1$ and $(\mathcal{S}_r)_r$ as above. Assume that there is $c > 0$ such that for any $R > 0$,*

$$(u \in \mathcal{S}_R) \wedge \big(u \geqslant 0 \text{ in } B_R(x_0)\big) \quad \text{implies} \quad \sup_{x \in B_{R/\theta}(x_0)} u \leqslant c \inf_{x \in B_{R/\theta}(x_0)} u. \tag{2}$$

Then there exist $\beta \in (0, 1)$ and $C > 0$ such that for any $R > 0$, any $u \in \mathcal{S}_R$ and any $x \in B_R(x_0)$

$$\big|u(x) - u(x_0)\big| \leqslant C \big\|u - u(x_0)\big\|_\infty \left(\frac{\mathrm{d}(x, x_0)}{R}\right)^\beta.$$

Because of this theorem, Harnack's inequality became very important in the study of various linear and nonlinear differential equations. The aim of this Note is to study the two theorems above for nonlocal operators.

2. Nonlocal operators

For $\alpha \in (0, 2)$, $u \in C_c^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we define

$$\Delta^{\alpha/2} u(x) = \mathcal{A}_{d,\alpha} \lim_{\varepsilon \to 0+} \int\limits_{|y-x|>\varepsilon} \frac{u(y) - u(x)}{|y - x|^{d+\alpha}} \, \mathrm{d}y, \tag{3}$$

where $\mathcal{A}_{d,\alpha} = \frac{\Gamma((d+\alpha)/2)}{2^{-\alpha}\pi^{d/2}|\Gamma(-\alpha/2)|}$. Using corresponding Poisson kernels one can easily prove the following version of Harnack's inequality for nonlocal operators:

Theorem 2.1. *(See [9].) There is a constant $c > 0$ such that for any function $u : \mathbb{R}^d \to \mathbb{R}$ with*

$$\Delta^{\alpha/2} u(x) = 0 \quad \text{for } x \in B_1, \tag{4}$$

$$u(x) \geqslant 0 \quad \text{for } x \in \mathbb{R}^d, \tag{5}$$

the following inequality holds:

$$u(x) \leqslant c u(y) \quad \text{for } x, y \in B_{1/2}.$$

Assuming nonnegativity of u in B_1 only is not sufficient as the following result shows.

Theorem 2.2. *(See [7].) There exists a function $u : \mathbb{R}^d \to \mathbb{R}$, $|u| \leqslant 1$, which is infinitely many times differentiable in B_1 and satisfies*

$$\Delta^{\alpha/2} u(x) = 0 \quad \text{for } x \in B_1,$$

$$u(x) > 0 \quad \text{for } x \in B_1 \setminus \{0\},$$

$$u(0) = 0.$$

The idea of the proof is to construct a function $g : \mathbb{R}^d \setminus B_1 \to \mathbb{R}$ of the form $g \equiv 1$ on $B_R \setminus B_1$, $g \equiv -1$ on $B_S \setminus B_R$ and $g \equiv 0$ on $\mathbb{R}^d \setminus B_S$. Then one defines u with the help of Poisson integrals.

One drawback of the formulation of Harnack's inequality in Theorem 2.1 is that it does not allow to apply Moser's approach in order deduce *a priori* estimates directly. Moreover, Theorem 2.1 does not generalize Harnack's result in the classical situation because of the assumption $u(x) \geqslant 0$ for $x \in \mathbb{R}^d$ for any $\alpha \in (0, 2)$. The following new formulation is equivalent to Theorem 2.1 but does not have these problems.

Theorem 2.3. *There is a constant $c > 0$ such that for any $u : \mathbb{R}^d \to \mathbb{R}$ satisfying*

$$\Delta^{\alpha/2} u(x) = 0, \quad x \in B_1,$$

the following estimate holds for any $x, y \in B_{1/2}$:

$$u(x) \leqslant c\big(u(y) + H^{\alpha/2}\big(B_1|u^-\big)(0)\big), \quad \textit{i.e.} \tag{6}$$

$$u(x) \leqslant cu(y) + c\alpha(2-\alpha) \int\limits_{\mathbb{R}^d \setminus B_1} \frac{u^-(z)}{(|z|^2 - 1^2)^{\alpha/2}|z|^d}\,\mathrm{d}z. \tag{7}$$

If, in addition, u is nonnegative in B_1, then

$$u(x) \leqslant cu(y) + c\alpha(2-\alpha) \int\limits_{\mathbb{R}^d \setminus B_1} \frac{u^-(z)}{|z|^{d+\alpha}}\,\mathrm{d}z. \tag{8}$$

Here, $H^{\alpha/2}(A|f)(x)$ stands for the Poisson integral with respect to the set A, the function f and the point $x \in A$. Obviously, Theorem 2.3 implies Theorem 2.1. Moreover, inequality (8) becomes Harnack's original result if $\alpha \to 2$. The proof of Theorem 2.3 is simple. One uses $u = u^+ - u^-$ and applies Theorem 2.1 to the function $x \mapsto H^{\alpha/2}(B_1|u^+)(x)$. This principle carries over to more general settings such as the cases treated in [1] and subsequent works or [3].

Lemma 2.4. *Let $(X, \mathcal{W})$ be a balayage space (cf. [5]) such that $1 \in \mathcal{W}$. Let V, W be open sets in X with $\overline{V} \subset W$. Let $c > 0$. Suppose that, for all $x, y \in V$ and $u \in \mathcal{H}_b^+(V)$,*

$$u(x) \leqslant cu(y). \tag{9}$$

Then for every $u \in \mathcal{H}_b(W)$

$$u(x) \leqslant cu(y) + c\int u^-\,\mathrm{d}\varepsilon_y^{V^c}. \tag{10}$$

Here, $\mathcal{H}_b(A)$ denotes the set of bounded functions which are harmonic in the Borel set A. Functions in $\mathcal{H}_b^+(A)$, in addition, are nonnegative. $\varepsilon_y^{V^c}$ denotes the harmonic measure of V with respect to $y \in V$.

Proof. Let $u \in \mathcal{H}_b(W)$. Then $u(x) = \varepsilon_x^{V^c}(u)$, $u(y) = \varepsilon_y^{V^c}(u)$ and hence

$$u(x) \leqslant \varepsilon_x^{V^c}\big(u^+\big) \leqslant c\varepsilon_y^{V^c}\big(u^+\big) = c\varepsilon_y^{V^c}\big(u + u^-\big) = cu(y) + c\int u^-\,\mathrm{d}\varepsilon_y^{V^c}.$$

The proof is complete. □

Note, that Theorem 2.3 provides a new formulation of Harnack's inequality in the case $\alpha = 2$, too.

Theorem 2.5. *There is a constant $c > 0$ such that for any harmonic function $u : B_1 \to \mathbb{R}$ the following estimate holds for any $x, y \in B_{1/2}$:*

$$u(x) \leqslant c\Bigg(u(y) + \int\limits_{\partial B_1} \frac{u^-(z)}{|z|^d}\,\mathrm{d}z\Bigg). \tag{11}$$

3. Applications

The strength of the above formulation of Harnack's inequality is that it allows us to extend Moser's approach. We use the set-up of the metric measure space (Ω, d, μ) from above. For $R > r > 0$, $x_0 \in \Omega$, set $A_{r,R}(x_0) = B_R(x_0) \setminus B_r(x_0)$. For every $r > 0$, let μ_r denote a finite measure on $\Omega \setminus B_r(x_0)$. We assume that, for some $\theta > 1$, some $x_0 \in \Omega$, and for every $j \in \mathbb{N}_0$,

$$\eta_j := \sup_{r>0} \mu_r\big(A_{\theta^j r, \theta^{j+1} r}(x_0)\big) < \infty \quad \text{and} \quad \limsup_{j\to\infty} \eta_j^{1/j} < 1. \tag{12}$$

Let us look at an example. Assume $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ satisfies for some $\alpha \in (0, 2)$, $\gamma > 0$, and $\kappa > 0, M \geqslant 0$:

$$\begin{aligned} &\kappa^{-1}|x-y|^{-d-\alpha} \leqslant k(x,y) \leqslant \kappa|x-y|^{-d-\alpha}, \quad |x-y| \leqslant 1, \\ &0 \leqslant k(x,y) \leqslant M|x-y|^{-d-\gamma}, \quad |x-y| \geqslant 1. \end{aligned} \tag{13}$$

For $r > 0$, A a Borel set in $\mathbb{R}^d \setminus B_r(x_0)$, set

$$\mu_r(A) = \Bigg(\int\limits_{\mathbb{R}^d \setminus B_r(x_0)} k(x_0, z)\,\mathrm{d}z\Bigg)^{-1} \int\limits_A k(x_0, y)\,\mathrm{d}y.$$

Then there is a constant $c(\kappa, M, \gamma) > 0$ such that for any $\theta > 1$, $r \in (0, 7/8)$ and $j \in \mathbb{N}$

$$\mu_r\big(A_{\theta^j r, \theta^{j+1} r}(x_0)\big) \leqslant \mu_r\big(\mathbb{R}^d \setminus B_{\theta^j r}(x_0)\big) \leqslant c(\kappa, M, \gamma) r^\alpha \big(\theta^j r\big)^{-\alpha} \leqslant c(\kappa, M, \gamma)\theta^{-j\alpha},$$

so that (12) is satisfied.

Here is an extension of Theorem 1.2:

Theorem 3.1. *Let $x_0 \in \Omega$, $(\mathcal{S}_r)_r$, $\theta > 1$ and $(\mu_r)_r$ be as above. Assume that there is $c > 0$ such that for any $R > 0$,*

$$(u \in \mathcal{S}_R) \wedge \big(u \geqslant 0 \text{ in } B_R(x_0)\big) \quad \text{implies} \quad \sup_{x \in B_{R/\theta}(x_0)} u \leqslant c \inf_{x \in B_{R/\theta}(x_0)} u + c \int_\Omega u^-(z)\,\mathrm{d}\mu_R(z). \tag{14}$$

Then there exist $\beta \in (0, 1)$ and $C > 0$ such that for any $R > 0$, any $u \in \mathcal{S}_R$ and any $x \in B_R(x_0)$

$$\big|u(x) - u(x_0)\big| \leqslant C\big\|u - u(x_0)\big\|_\infty \left(\frac{\mathrm{d}(x, x_0)}{R}\right)^\beta. \tag{15}$$

The proof of this result follows Moser's ideas very closely.

Let us finally give an application to nonlocal Dirichlet forms. Assume $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ satisfies for some $\alpha \in (0, 2)$ and positive reals c, γ, M:

$$\begin{aligned} &c^{-1}\mathcal{A}_{d,\alpha}|x - y|^{-d-\alpha} \leqslant k(x, y) \leqslant c\mathcal{A}_{d,\alpha}|x - y|^{-d-\alpha}, \quad |x - y| \leqslant 1, \\ &0 \leqslant k(x, y) \leqslant M\mathcal{A}_{d,\alpha}|x - y|^{-d-\gamma}, \quad |x - y| > 1. \end{aligned} \tag{16}$$

Under these assumptions and

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(u(y) - u(x)\big)\big(v(y) - v(x)\big)k(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

the tuple $(\mathcal{E}, H^{\alpha/2}(\mathbb{R}^d))$ is a regular Dirichlet-form. For an open bounded domain Ω, set $D_\Omega(\mathcal{E}) = L^\infty(\mathbb{R}^d) \cap H^{\alpha/2}_{\mathrm{loc}}(\Omega)$.

A priori estimates in Hölder spaces for similar Dirichlet forms have been obtained in [8,2,4] and other works. The methods used there do not give uniform estimates for $\alpha \to 2$. With the help of Theorem 3.1 it is possible to establish a uniform result:

Theorem 3.2. *There are $\beta \in (0, 1)$ and $C > 0$ such that for any $\alpha \in (\alpha_0, 2)$, $u \in D_{B_1}(\mathcal{E})$ with $\mathcal{E}(u, \phi) = 0$ for any $\phi \in C_c^\infty(B_1)$ the following inequality holds for almost any $x, y \in B_{1/2}$:*

$$\big|u(x) - u(y)\big| \leqslant C\|u\|_\infty\,\mathrm{d}(x, y)^\beta. \tag{17}$$

As in Moser's original result one can use test functions of the form $\phi = \tau^2\,\mathrm{sign}(p)u^p$ where τ is a smooth cut-off function. For $p = -1$ one deduces that $\log u$ belongs to $\mathrm{BMO}(B_{3/4})$ which is a key step in the proof.

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References

[1] R.F. Bass, D.A. Levin, Harnack inequalities for jump processes, Potential Anal. 17 (4) (2002) 375–388.
[2] R.F. Bass, D.A. Levin, Transition probabilities for symmetric jump processes, Trans. Amer. Math. Soc. 354 (7) (2002) 2933–2953 (electronic).
[3] K. Bogdan, P. Sztonyk, Estimates of potential kernel and Harnack's inequality for anisotropic fractional Laplacian, Stud. Math. 181 (2) (2007) 101–123.
[4] Zh.-Qing Chen, T. Kumagai, Heat kernel estimates for stable-like processes on d-sets, Stochastic Process. Appl. 108 (1) (2003) 27–62.
[5] W. Hansen, J. Bliedtner, Potential Theory: An Analytic and Probabilistic Approach to Balayage (Universitext), Springer, Berlin, 1986.
[6] C.-G. Axel Harnack, Die Grundlagen der Theorie des logarithmischen Potentiales und der eindeutigen Potentialfunktion in der Ebene, Teubner, Leipzig, 1887, see also The Cornell Library Historical Mathematics Monographs online.
[7] M. Kassmann, The classical Harnack inequality fails for nonlocal operators, Preprint No. 360, Sonderforschungsbereich 611, Universität Bonn, 2007.
[8] T. Komatsu, Uniform estimates for fundamental solutions associated with non-local Dirichlet forms, Osaka J. Math. 32 (4) (1995) 833–860.
[9] N.S. Landkof, Foundations of Modern Potential Theory, Springer-Verlag, New York, 1972, translated from the Russian by A.P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.
[10] J. Moser, On Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math. 14 (1961) 577–591.