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Partial Differential Equations

Anisotropic entire large solutions

Solutions entières, explosives et anisotropes

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A B S T R A C T

Given $q \in (0,1]$, we construct nonradial entire large solutions to the equation $\Delta u = u^q$ in $\mathbb{R}^N$.

R É S U M É

Pour $q \in (0,1]$, nous construisons des solutions globales, explosives, et non radiales de l'équation $\Delta u = u^q$ dans $\mathbb{R}^N$.

1. Introduction

The starting point of this work is the following question, due to H. Brezis:

Question 1.1. (See [2].) For $N \geqslant 2$ and $q \in (0,1]$, consider the equation

$$\Delta u = u^q - u^{-q-2} \quad \text{in } \mathbb{R}^N. \tag{1}$$

Is it true that all (smooth positive) solutions such that $u(x) \to +\infty$ as $|x| \to +\infty$ are radial about some point in $\mathbb{R}^N$?

The main feature of the nonlinearity in the right-hand side of (1) is its sublinear growth at infinity: nontrivial solutions only exist when $q \leqslant 1$, as follows from the classical works of J.B. Keller [6] and R. Osserman [7] (see also [3]).

Solutions blowing up at infinity are often called entire large solutions in the literature (ELS, for short). Indications that there exist nonradial ELS to (1) abound: S. Taliaferro [8] observed that given a unit vector $\alpha = (\alpha_1, \dots, \alpha_N)$, the function

$$u(x) = \cosh(\alpha_1 x_1) \dots \cosh(\alpha_N x_N)$$

is a nonradial ELS to

$$\Delta u = u \quad \text{in } \mathbb{R}^N.$$

Using separation of variables and standard bifurcation theory, M.F. Bidaut-Véron and P. Grillot [1] constructed nonradial ELS to the equation

$$\Delta u = u^q \quad \text{in } \mathbb{R}^N, \tag{2}$$

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of the form

$$u(x) = r^{\frac{2}{1-q}} v(\theta), \quad \text{where } r = |x| \text{ and } \theta = x/r, \tag{3}$$

for specific values of $q \in (0, 1]$. As demonstrated by the authors of [1], solutions of the form (3) with v not a constant, cannot exist for all values of $q \in (0, 1]$. This does not rule out the possibility of solutions of a different form. In this note, I construct nonradial solutions of a different nature, which exist for all q's.

Theorem 1.2. *Assume $N \geqslant 2$ and $q \in (0, 1]$. There exists a positive smooth solution to* (2) *which is not radial about any point in $\mathbb{R}^N$ and such that*

$$\lim_{|x| \to +\infty} u(x) = +\infty.$$

As a simple extension, I obtain a negative answer to Brezis's question:

Theorem 1.3. *Assume $N \geqslant 2$, let $a \in \mathbb{R}$, and let f be a C^2 real valued function defined on the interval $[a, +\infty)$. Assume that $f \not\equiv 0$ is nonnegative, nondecreasing, concave, and such that for some constant $C > 0$,*

$$C f''(t) \leqslant f''(2t) \leqslant \frac{1}{C} f''(t), \quad \textit{for all } t \geqslant a. \tag{4}$$

Then, there exists a C^2 solution to

$$\Delta u = f(u) \quad \textit{in } \mathbb{R}^N \tag{5}$$

which is not radial about any point in $\mathbb{R}^N$ and such that

$$\lim_{|x| \to +\infty} u(x) = +\infty.$$

Remark 1.4. As mentioned earlier, there are no solutions to (5) whenever f is positive and satisfies the Keller–Osserman condition

$$\int^{+\infty} \frac{\mathrm{d}t}{\sqrt{2F(t)}} < +\infty, \tag{6}$$

where $F' = f$. However, the equation

$$\Delta u = \rho\big(|x|\big) f(u) \quad \text{in } \mathbb{R}^N$$

can be solved for f satisfying (6), provided ρ decays fast enough at infinity. In a forthcoming paper [4], we shall demonstrate that radial symmetry does hold for such equations. So, it would be interesting to know whether Theorem 1.3 remains true for any positive nonlinearity which fails the Keller–Osserman condition. In that respect, the careful reader will see that the proof I present below cannot work for slightly superlinear nonlinearities, such as $f(t) = t \ln(t)^q$, $q < 2$. Such nonlinearities fail (6).

2. Proof of Theorem 1.2

From the discussion in the introduction, we can restrict to the case $q \in (0, 1)$. Let u_0 be the unique radial solution of (2) such that $u_0(0) = 1$ and $u_0'(0) = 0$. It is well known (see e.g. [5]) that u_0 is globally defined and blows up at infinity at a fixed rate

$$\lim_{r \to +\infty} r^{-\alpha} u_0(r) = L, \tag{7}$$

where $\alpha = \frac{2}{1-q}$ and $L = [\alpha(\alpha + N - 2)]^{-\frac{1}{q-1}} > 0$. Integrating (2), we have

$$\frac{\mathrm{d}u_0}{\mathrm{d}r} = r^{1-N} \int_0^r t^{N-1} u_0^q(t)\, \mathrm{d}t,$$

from which it easily follows that

$$\lim_{r \to +\infty} r^{-\alpha+1} \frac{\mathrm{d}u_0}{\mathrm{d}r} = L\alpha. \tag{8}$$

We are going to construct a solution of the form

$$u = u_0 + \epsilon(u_1 + v), \tag{9}$$

where

$$u_1 = \frac{\partial u_0}{\partial x_1}$$

solves the linearized equation

$$-\Delta u_1 + q u_0^{q-1} u_1 = 0 \quad \text{in } \mathbb{R}^N$$

and by (7)–(8),

$$u_1 = \frac{\partial u_0}{\partial x_1} = \frac{\mathrm{d}u_0}{\mathrm{d}r}\frac{x_1}{r} = o(u_0), \quad \text{as } |x| \to +\infty.$$

Eq. (2) is then equivalent to

$$\begin{aligned} -\Delta v + q u_0^{q-1} v &= -\frac{1}{\epsilon}\Big(\big(u_0 + \epsilon(u_1+v)\big)^q - u_0^q - q u_0^{q-1}\epsilon(u_1+v)\Big) \\ &= \epsilon\frac{q(1-q)}{2}\big(u_0 + t\epsilon(u_1+v)\big)^{q-2}(u_1+v)^2 \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{10}$$

where $t = t(u_0, u_1, v, \epsilon) \in [0, 1]$.

To solve (10), we observe that $\underline{v} = 0$ is a subsolution, while $\bar{v} = u_0^q + 1$ solves

$$-\Delta\bar{v} + q u_0^{q-1}\bar{v} = q(1-q)u_0^{q-2}\left(\frac{\mathrm{d}u_0}{\mathrm{d}r}\right)^2 + q u_0^{q-1}.$$

It follows that for ϵ sufficiently small, $\bar{v}$ is a supersolution of (10).

Hence, for each $R > 1$, we may find (e.g. by monotone iteration) a function $v = v_R$ such that (10) holds in B_R, $v_R = 0$ on ∂B_R, and

$$0 = \underline{v} \leqslant v_R \leqslant \bar{v} = u_0^q + 1 \quad \text{in } B_R.$$

By elliptic regularity, a sequence (v_{R_n}) converges in $C^2_{loc}(\mathbb{R}^N)$ to a solution v of (10) such that

$$0 \leqslant v \leqslant u_0^q + 1 \quad \text{in } \mathbb{R}^N.$$

In particular, $v = o(u_1)$, when $x = (x_1, 0)$ and $x_1 \to \pm\infty$. So the function

$$u = u_0 + \epsilon(u_1 + v)$$

solves (2) and cannot be radial about any point in $\mathbb{R}^N$. $\square$

3. Proof of Theorem 1.3

If f is constant, then a constant multiple of $u(x) = |x|^2 + x_1$ is the desired nonradial solution. Since f is nondecreasing and concave, we may now assume that $f' > 0$. Take the radial solution u_0 to (5) with initial values $u_0(0) = b > \max(a, 0)$, $u_0'(0) = 0$. As follows from the work of J.B. Keller [6], u_0 is globally defined, nondecreasing, and blows up at infinity. Furthermore, the following estimate holds

$$\frac{1}{N}\sqrt{2F(u_0)} \leqslant \frac{\mathrm{d}u_0}{\mathrm{d}r} \leqslant \sqrt{2F(u_0)}, \tag{11}$$

where $F(t) = \int_b^t f(s)\,\mathrm{d}s$. We look again for a solution of the form $u = u_0 + \epsilon(u_1 + v)$, where $u_1 = \partial u_0/\partial x_1$. That is, we seek v solving

$$\begin{aligned} -\Delta v + f'(u_0)v &= -\frac{1}{\epsilon}\big[f\big(u_0 + \epsilon(u_1+v)\big) - f(u_0) - f'(u_0)\epsilon(u_1+v)\big] \\ &= -\frac{\epsilon}{2}f''\big(u_0 + t\epsilon(u_1+v)\big)(u_1+v)^2 \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{12}$$

where $t = t(u_0, u_1, v, \epsilon) \in [0, 1]$. Since f is concave, $\underline{v} = 0$ is a subsolution of (12). Now let $\bar{v} = f(u_0) + 1$. Then,

$$-\Delta\bar{v} + f'(u_0)\bar{v} = -f''(u_0)\left(\frac{\mathrm{d}u_0}{\mathrm{d}r}\right)^2 + f'(u_0).$$

By L'Hôpital's rule,

$$\lim_{+\infty} \frac{f^2}{2F} = \lim_{+\infty} f' =: l \in [0, +\infty). \tag{13}$$

Using (11), we deduce that

$$(u_1 + \bar{v})^2 \leqslant C\left(\frac{\mathrm{d}u_0}{\mathrm{d}r}\right)^2 \quad \text{for } |x| \geqslant 1. \tag{14}$$

By L'Hôpital's rule again,

$$\lim_{t\to+\infty} \frac{2F(t)}{t^2} = \lim_{+\infty} f' = l \in [0, +\infty).$$

Using (11) again, we deduce that

$$\frac{\mathrm{d}u_0}{\mathrm{d}r} \leqslant Cu_0. \tag{15}$$

Collecting (14) and (15), it follows from our assumption (4) that

$$-f''\bigl(u_0 + t\epsilon(u_1 + \bar{v})\bigr) \leqslant -Cf''(u_0). \tag{16}$$

From (14) and (16), we conclude that $\bar{v}$ is a supersolution to (12), provided $\epsilon > 0$ is chosen sufficiently small. By monotone iteration, we obtain a solution $u = u_0 + \epsilon(u_1 + v)$ of (5), where

$$0 \leqslant v \leqslant f(u_0) + 1. \tag{17}$$

In case $l = 0$, (17), (13), and (11) imply that $v = o(u_1)$ when $x = (x_1, 0)$ and $x_1 \to \pm\infty$. So, u cannot be radial about any point in $\mathbb{R}^N$.

In case $l > 0$, we may always assume that $l < \frac{1}{N^2}$, using the change of independent variable $y = \lambda x$ if necessary. It follows from (17), (13), and (11) that

$$\limsup_{|x_1|\to\infty} \frac{v(x_1, 0)}{|u_1(x_1, 0)|} < 1,$$

and so again, u cannot be radial about any point in $\mathbb{R}^N$. $\square$

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