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Partial Differential Equations

On the use of T-coercivity to study the interior transmission eigenvalue problem

Utilisation de la T-coercivité pour l'étude du problème de transmission intérieur

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ABSTRACT

In this Note, we investigate the so-called interior transmission problem using the T-coercivity approach. In particular, we prove that this problem, which appears when one is interested in the reconstruction of the support of an inclusion embedded in a homogeneous medium, is of Fredholm type and that so-called transmission eigenvalues form at most a discrete set. Our approach treats cases where the difference between the inclusion index and the background index can change sign, which are not covered by other techniques that can be found in the literature. We also provide Faber–Krahn type inequalities associated with this general case.

RÉSUMÉ

Dans cette Note, nous utilisons la méthode de la T-coercivité pour étudier le problème de transmission intérieur. En particulier, nous prouvons que ce problème, qui apparaît lorsqu'on cherche à reconstruire le support d'une inclusion dans un milieu homogène, est de type Fredholm et que l'ensemble de ses valeurs propres est discret. Notre approche permet de traiter des cas pour lesquels la différence entre l'indice de l'inclusion et celui du milieu environnant change de signe, cas qui n'étaient pas couverts par les techniques classiques de la littérature. Nous donnons également des inégalités de type Faber–Krahn dans ces situations.

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Le problème de transmission intérieur (cf. (2) ci-dessous) joue un rôle important dans la théorie des problèmes inverses de diffraction par une hétérogénéité (voir [3,6]). L'opérateur associé à la formulation variationnelle « naturelle » de ce problème n'est pas elliptique en raison d'un changement de signe dans sa partie principale. En électromagnétisme, on rencontre une difficulté analogue dans l'étude du problème de transmission entre un diélectrique classique et un métamatériau négatif en régime harmonique. Pour y faire face, on peut utiliser la technique de la T-coercivité introduite dans [2]. L'idée consiste, dans les formulations variationnelles, à tester non pas directement contre le champ physique, mais contre une

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transformation géométrique simple de ce champ. Ceci permet de retrouver une certaine positivité de la partie principale de l'opérateur sous-jacent.

Dans cette Note, nous nous proposons d'adapter cette méthode de la T-coercivité pour étudier le problème de transmission intérieur. Nous montrons que ce dernier est bien posé au sens de Fredholm et que l'ensemble des valeurs propres de transmission est discret dénombrable. Notre résultat s'applique à des situations qui n'avaient pas encore pu être traitées par les techniques usuelles (voir [3,5,8]), comme par exemple le cas où l'indice de réfraction prend des valeurs supérieures et inférieures à 1 sur le domaine. Plus précisément, avec les notations du §1, nous prouvons les théorèmes suivants :

Théorème 0.1.

- *Si $A(x) \leqslant A^\star Id < Id$ ou si $Id < A_\star Id \leqslant A(x)$ p.p. dans un voisinage de ∂D, alors pour tout $k \in \mathbb{C}$, l'opérateur $\mathcal{A}_k$ associé au problème de transmission intérieur est Fredholm de X dans X.*
- *Si $A(x) \leqslant A^\star Id < Id$ et $n(x) \leqslant n^\star < 1$ ou si $Id < A_\star Id \leqslant A(x)$ et $1 < n_\star \leqslant n(x)$ p.p. dans un voisinage de ∂D, alors l'ensemble des valeurs propres de transmission est discret dénombrable dans $\mathbb{C}$. En outre, il existe deux constantes positives ρ et δ telles que si $k \in \mathbb{C}$ vérifie $|k| > \rho$ et $|\Re e\, k| < \delta|\Im m\, k|$, alors k n'est pas valeur propre de transmission.*

Théorème 0.2. *Supposons $\int_D (n-1) \neq 0$.*

- *Si $A_+ < 1$ alors l'ensemble des valeurs propres de transmission est discret dénombrable dans $\mathbb{C}$. De plus, la valeur propre de transmission non triviale de plus petit module k_1 vérifie l'estimation de type Faber–Krahn $|k_1|^2 \geqslant (A_-(1-\sqrt{A_+}))/(C_P \max(n_+, 1)(1+\sqrt{n_+}))$, avec C_P définie en (10).*
- *Si $1 < A_-$ alors l'ensemble des valeurs propres de transmission est discret dénombrable dans $\mathbb{C}$. De plus, la valeur propre de transmission non triviale de plus petit module k_1 vérifie l'estimation de type Faber–Krahn $|k_1|^2 \geqslant (1-1/\sqrt{A_-})/(C_P \max(n_+, 1)(1+1/\sqrt{n_-}))$, avec C_P définie en (10).*

Lorsque $A - Id$ change de signe dans un voisinage de la frontière ou pire encore, s'y annule, nous pensons qu'il existe des géométries et des valeurs de A telles que l'opérateur associé au problème de transmission ne soit pas Fredholm dans X à cause de l'apparition de singularités qui sortent de H^1. Pour tout $k \in \mathbb{C}$, l'opérateur $\mathcal{A}_k$ n'est alors plus à image fermée (dans [1], les auteurs démontrent ce résultat pour le problème de transmission entre un diélectrique et un métamatériau négatif). Nous ne nous intéressons pas à cette situation dans ce travail. La technique de la T-coercivité semble également utilisable pour étudier le problème de transmission intérieur associé aux équations de Maxwell. La question de l'existence de valeurs propres de transmission réelles revêt une grande importance pour les applications. Cependant, à l'heure actuelle, nous ne sommes pas en mesure d'y répondre par l'approche T-coercivité.

1. Introduction

The interior transmission problem plays an important role in the inverse scattering theory for inhomogeneous media (see [3] for a presentation of the anisotropic case). The operator associated with the variational formulation of the problem is not elliptic because of a sign changing in its principal part. A similar difficulty appears in the study of the wave transmission problem between a classical dielectric and a negative metamaterial in harmonic regime. Recently, some authors have introduced the T-coercivity method (see [2]) to deal with that last problem.

In this work, we develop the T-coercivity approach for the interior transmission problem in the scalar case when the contrast in the scattering medium occurs in two independent functions A and n (see (2)) which are respectively equal to Id and 1 for the reference medium. In particular, we prove that this problem is of Fredholm type and that the set of interior transmission eigenvalues is discrete. Our result applies to situations for which $A - Id$ and $n - 1$ are positive or negative in a neighbourhood of the boundary and change sign inside the domain. That latter case was not covered by the usual techniques (see [3,5,8]) and has been an open problem for a long time. We also give estimates for the first eigenvalue in the case when A is smaller or greater than one all over the domain. When the sign of $A - Id$ changes (or worse, when $A - Id$ reduces to zero) in the neighbourhood of the boundary, we think there are geometries and values of A for which the interior transmission problem is not Fredholm in H^1 because of the apparition of "strong" singularities (that result is proved in [1] for the transmission problem between a dielectric and a negative metamaterial). This situation is excluded here. The T-coercivity approach seems to be extendable to deal with the interior transmission problem for Maxwell's equations. Up to now, we have not been able to prove results of existence of real transmission eigenvalues with the T-coercivity technique. Indeed, the equivalent formulation of problem (3) we consider, which presents a useful property of positivity, is no longer symmetric: that prevents using the classical *min–max* arguments (see [4,6]). Hence, the question of existence of real transmission eigenvalues when $A - Id$ or $n - 1$ change sign is still an open question.

2. Setting of the problem

Consider D a bounded connected domain of $\mathbb{R}^3$, with Lipschitz boundary ∂D and denote by ν the outward unit normal. Let $A \in L^\infty(D, \mathbb{R}^{3\times 3})$ be a matrix valued function such that $A(x)$ is symmetric for almost all $x \in D$. $n \in L^\infty(D, \mathbb{R})$ will be a scalar real valued function. We suppose that

$$\begin{aligned} &A_- := \inf_{x\in D} \inf_{\xi\in\mathbb{R}^3, |\xi|=1} \big(\xi \cdot A(x)\xi\big) > 0; \qquad A_+ := \sup_{x\in D} \sup_{\xi\in\mathbb{R}^3, |\xi|=1} \big(\xi \cdot A(x)\xi\big) < \infty; \\ &n_- := \inf_{x\in D} n(x) > 0 \quad \text{and} \quad n_+ := \sup_{x\in D} n(x) < \infty. \end{aligned} \tag{1}$$

If $\mathcal{O}$ is an open subset of $\mathbb{R}^3$, we denote by $(\cdot,\cdot)_{\mathcal{O}}$ the Hermitian scalar products of $L^2(\mathcal{O})$ and $(L^2(\mathcal{O}))^3$, and by $\|\cdot\|_{\mathcal{O}}$ the associated norms. The transmission eigenvalue problem reads:

$$\left| \begin{array}{ll} \text{Find } (u,w) \in H^1(D) \times H^1(D) \text{ such that:} & \\ \operatorname{div}(A\nabla u) + k^2 n u = 0 & \text{in } D \\ \Delta w + k^2 w = 0 & \text{in } D \\ u - w = 0 & \text{on } \partial D \\ \nu \cdot A\nabla u - \nu \cdot \nabla w = 0 & \text{on } \partial D. \end{array} \right. \tag{2}$$

Values of $k \in \mathbb{C}$ for which problem (2) has a nontrivial solution (u,w) are called transmission eigenvalues. One can show that (u,w) satisfies problem (2) if and only if (u,w) satisfies the problem

$$\left| \begin{array}{l} \text{Find } (u,w) \in X \text{ such that, for all } \big(u',w'\big) \in X, \\ a_k\big((u,w),\big(u',w'\big)\big) := \big(A\nabla u, \nabla u'\big)_D - \big(\nabla w, \nabla w'\big)_D - k^2\big(\big(nu,u'\big)_D - \big(w,w'\big)_D\big) = 0, \end{array} \right. \tag{3}$$

with $X = \{(u,w) \in H^1(D) \times H^1(D) \mid u - w \in H_0^1(D)\}$. With Riesz representation theorem, we define the operator $\mathcal{A}_k$ from X to X such that, for all $((u,w),(u',w')) \in X \times X$, $(\mathcal{A}_k(u,w),(u',w'))_{H^1(D)\times H^1(D)} = a_k((u,w),(u',w'))$. This eigenvalue problem differs from classical ones because a_k is not coercive on X neither "coercive + compact".

3. Outline: the *T*-coercivity method

For the sake of clarity, we present the technique in the simple case: $A_+ < 1$ and $n_+ < 1$. The idea is to consider an equivalent formulation of (3) where a_k is replaced by a_k^T defined by

$$a_k^T\big((u,w),\big(u',w'\big)\big) := a_k\big((u,w), T\big(u',w'\big)\big), \quad \forall\big((u,w),\big(u',w'\big)\big) \in X \times X, \tag{4}$$

T being an *ad hoc* isomorphism of X. Indeed, $(u,w) \in X$ satisfies $a_k((u,w),(u',w')) = 0$ for all $(u',w') \in X$ if, and only if, it satisfies $a_k^T((u,w),(u',w')) = 0$ for all $(u',w') \in X$. In the present case, $T(u,w) := (u - 2w, -w)$ (T is an isomorphism since $T^2 = Id$). Using Young's inequality, one has for $k = i\kappa$ with $\kappa \in \mathbb{R}^*$, $\forall\alpha,\beta > 0$, $\forall(u,w) \in X$,

$$\begin{aligned} \big|a_k^T\big((u,w),(u,w)\big)\big| &= \big|(A\nabla u,\nabla u)_D + (\nabla w,\nabla w)_D - 2(A\nabla u,\nabla w)_D + \kappa^2\big((nu,u)_D + (w,w)_D - 2(nu,w)_D\big)\big| \\ &\geqslant (A\nabla u,\nabla u)_D + (\nabla w,\nabla w)_D + \kappa^2\big((nu,u)_D + (w,w)_D\big) - 2\big|(A\nabla u,\nabla w)_D\big| - 2\kappa^2\big|(nu,w)_D\big| \\ &\geqslant \big((1-\alpha)A\nabla u,\nabla u\big)_D + \big(\big(1-\alpha^{-1}A_+\big)\nabla w,\nabla w\big)_D \\ &\quad + \kappa^2\big(\big((1-\beta)nu,u\big)_D + \big(\big(1-\beta^{-1}n_+\big)w,w\big)_D\big). \end{aligned} \tag{5}$$

Taking α and β such that $A_+ < \alpha < 1$ and $n_+ < \beta < 1$, this estimate proves that a_k^T is coercive over X. Using Lax–Milgram theorem and since T is an isomorphism of X, one deduces that $\mathcal{A}_k$ is an isomorphism of X for $k = i\kappa$ with $\kappa \in \mathbb{R}^*$. Besides, for a general $k \in \mathbb{C}$, the operator $\mathcal{A}_k$ is a compact perturbation of an isomorphism of X since the embedding of X in $L^2(D) \times L^2(D)$ is compact.

4. Case $A \leqslant A^\star Id < Id$ on a neighbourhood of ∂D

We suppose in this paragraph that there exist a neighbourhood $\mathcal{V}$ of ∂D and a real constant $A^\star$ such that, in the sense of symmetric matrices (see (1)), $A(x) \leqslant A^\star Id < Id$ a.e. on $D \cap \mathcal{V}$.

Lemma 4.1. *If $A(x) \leqslant A^\star Id < Id$ and $n(x) \leqslant n^\star < 1$ a.e. on $D \cap \mathcal{V}$, then there exists $k = i\kappa$, with $\kappa \in \mathbb{R}$, such that the operator $\mathcal{A}_k$ is an isomorphism of X.*

Remark 1. Notice there is no assumption on the sign of $A - Id$ and $n - 1$ on $D \setminus \overline{\mathcal{V}}$.

Proof. Introduce $\chi \in \mathcal{C}^\infty(\overline{D})$ a cut off function equal to 1 in a neighbourhood of ∂D, with support in $\mathcal{V} \cap D$ and such that $0 \leqslant \chi \leqslant 1$, and consider the isomorphism ($T^2 = Id$) of X defined by $T(u,w) = (u - 2\chi w, -w)$. Let us prove that $a^T_{i\kappa}$ defined in (4) is coercive for some $\kappa \in \mathbb{R}$. For all $(u,w) \in X$, one has,

$$\begin{aligned}\left|a^T_{i\kappa}\big((u,w),(u,w)\big)\right| = \big|&(A\nabla u, \nabla u)_D + (\nabla w, \nabla w)_D - 2\big(A\nabla u, \nabla(\chi w)\big)_D \\ &+ \kappa^2\big((nu,u)_D + (w,w)_D - 2(nu, \chi w)_D\big)\big|.\end{aligned} \tag{6}$$

Using Young's inequality, one can write, for all $\alpha > 0$, $\beta > 0$, $\eta > 0$,

$$\begin{aligned}2\left|\big(A\nabla u, \nabla(\chi w)\big)_D\right| &\leqslant 2\left|(\chi A\nabla u, \nabla w)_{\mathcal{V}}\right| + 2\left|\big(A\nabla u, \nabla(\chi)w\big)_{\mathcal{V}}\right| \\ &\leqslant \eta(A\nabla u, \nabla u)_{\mathcal{V}} + \eta^{-1}(A\nabla w, \nabla w)_{\mathcal{V}} \\ &\quad + \alpha(A\nabla u, \nabla u)_{\mathcal{V}} + \alpha^{-1}\big(A\nabla(\chi)w, \nabla(\chi)w\big)_{\mathcal{V}} \quad \text{and} \\ 2\left|(nu, \chi w)_D\right| &\leqslant \beta(nu,u)_{\mathcal{V}} + \beta^{-1}(nw,w)_{\mathcal{V}}.\end{aligned} \tag{7}$$

Plugging (7) in (6), one obtains

$$\begin{aligned}\left|a^T_{i\kappa}\big((u,w),(u,w)\big)\right| \geqslant\ &(A\nabla u, \nabla u)_{D\setminus\overline{\mathcal{V}}} + (\nabla w, \nabla w)_{D\setminus\overline{\mathcal{V}}} + \kappa^2\big((nu,u)_{D\setminus\overline{\mathcal{V}}} + (w,w)_{D\setminus\overline{\mathcal{V}}}\big) \\ &+ \big((1-\eta-\alpha)A\nabla u, \nabla u\big)_{\mathcal{V}} + \big((Id - \eta^{-1}A)\nabla w, \nabla w\big)_{\mathcal{V}} \\ &+ \kappa^2\big((1-\beta)nu, u\big)_{\mathcal{V}} + \Big(\big(\kappa^2\big(1-\beta^{-1}n\big) - \sup_{\mathcal{V}}|\nabla\chi|^2 A^\star\alpha^{-1}\big)w, w\Big)_{\mathcal{V}}.\end{aligned}$$

Taking η, α and β such that $A^\star < \eta < 1$, $n^\star < \beta < 1$ and $0 < \alpha < 1 - \eta$, we obtain the coercivity of $a^T_{i\kappa}$ for κ large enough. □

Consider $\kappa_0 \in \mathbb{R}$ such that $\mathcal{A}_{i\kappa_0}$ is an isomorphism of X. Since $\mathcal{A}_k - \mathcal{A}_{i\kappa_0}$ is a compact operator of $L^2(D) \times L^2(D)$ for all $k \in \mathbb{C}$, using the analytic Fredholm theory, we deduce the following theorem:

Theorem 4.2. *If $A(x) \leqslant A^\star Id < Id$ and $n(x) \leqslant n^\star < 1$ a.e. on $D \cap \mathcal{V}$, then the set of transmission eigenvalues is discrete in $\mathbb{C}$.*

As an other direct consequence of Lemma 4.1, one has the following proposition:

Proposition 4.1. *Suppose only $A(x) \leqslant A^\star Id < Id$ a.e. on $D \cap \mathcal{V}$. Then for all $k \in \mathbb{C}$, the operator $\mathcal{A}_k$ is a Fredholm operator of X.*

Now, let us state a result of localisation of the transmission eigenvalues. In [7], the authors give a more precise statement for the bilaplacian problem.

Theorem 4.3. *If $A(x) \leqslant A^\star Id < Id$ and $n(x) \leqslant n^\star < 1$ a.e. on $D \cap \mathcal{V}$, there exist two positive constants ρ and δ such that if $k \in \mathbb{C}$ satisfies $|k| > \rho$ and $|\Re e\, k| < \delta|\Im m\, k|$, then k is not a transmission eigenvalue.*

Proof. Consider again the isomorphism T defined by $T(u,w) = (u - 2\chi w, -w)$. Lemma 4.1 proves that for $\kappa \in \mathbb{R}$ with $|\kappa|$ large enough, there holds the estimate

$$\left|a^T_{i\kappa}\big((u,w),(u,w)\big)\right| \geqslant C_1\big(\|u\|^2_{H^1(D)} + \|w\|^2_{H^1(D)}\big) + C_2\kappa^2\big(\|u\|^2_D + \|w\|^2_D\big), \tag{8}$$

where the constants $C_1, C_2 > 0$ are independent of κ. Take now $k = i\kappa e^{i\theta}$ with $\theta \in [-\pi/2; \pi/2]$. One has

$$\left|a^T_k\big((u,w),(u,w)\big) - a^T_{i\kappa}\big((u,w),(u,w)\big)\right| \leqslant C_3\left|1 - e^{2i\theta}\right|\kappa^2\big(\|u\|^2_D + \|w\|^2_D\big), \tag{9}$$

with $C_3 > 0$ independent of κ. Combining (8) and (9), one finds

$$\begin{aligned}\left|a^T_k\big((u,w),(u,w)\big)\right| &\geqslant \left|a^T_{i\kappa}\big((u,w),(u,w)\big)\right| - C_3\kappa^2\left|1 - e^{2i\theta}\right|\big(\|u\|^2_D + \|w\|^2_D\big) \\ &\geqslant C_1\big(\|u\|^2_{H^1(D)} + \|w\|^2_{H^1(D)}\big) + \big(C_2 - C_3\left|1 - e^{2i\theta}\right|\big)\kappa^2\big(\|u\|^2_D + \|w\|^2_D\big).\end{aligned}$$

Choosing θ small enough, to have for example $C_3|1 - e^{2i\theta}| \leqslant C_2/2$, one obtains the result. □

With a stronger assumption on A, we can weaken the condition on n. Taking $u' = w' = 1$ in (3), we first notice that the interior transmission eigenvectors (u,w) (i.e. the eigenvectors of problem (2)) satisfy $k^2 \int_D nu - w = 0$. That leads us to introduce the space $Y := \{(u,w) \in X \mid \int_D nu - w = 0\}$. Now, suppose $\int_D (n-1) \neq 0$. Arguing by contradiction, one can prove the existence of a Poincaré constant $C_P > 0$ (which depends on D and also on n through Y) such that

$$\|u\|_D^2 + \|w\|_D^2 \leqslant C_P\big(\|\nabla u\|_D^2 + \|\nabla w\|_D^2\big), \quad \forall (u,w) \in Y. \tag{10}$$

Moreover, one can check that $k \neq 0$ is a transmission eigenvalue if and only if there exists a nontrivial element $(u,w) \in Y$ such that $a_k((u,w),(u',w')) = 0$ for all $(u',w') \in Y$.

Theorem 4.4. *Suppose $\int_D (n-1) \neq 0$ and $A_+ < 1$. Then the set of transmission eigenvalues is discrete in $\mathbb{C}$. Besides, the eigenvalue of smallest magnitude k_1 such that $k_1 \neq 0$ satisfies the Faber–Krahn type estimate $|k_1|^2 \geqslant (A_-(1-\sqrt{A_+}))/(C_P \max(n_+,1)\,(1+\sqrt{n_+}))$ with C_P defined in* (10)*.*

Proof. Denote $\lambda(w) := 2\int_D (n-1)w / \int_D (n-1)$ and consider the isomorphism of Y defined by $T(u,w) = (u - 2w + \lambda(w), -w + \lambda(w))$ (notice that $\lambda(\lambda(w)) = 2\lambda(w)$ so $T^2 = Id$). For all $(u,w) \in Y$, one has

$$\begin{aligned}\big|a_k^T\big((u,w),(u,w)\big)\big| &= \big|(A\nabla u,\nabla u)_D + (\nabla w,\nabla w)_D - 2(A\nabla u,\nabla w)_D - k^2\big((nu,u)_D + (w,w)_D - 2(nu,w)_D\big)\big| \\ &\geqslant (A\nabla u,\nabla u)_D + (\nabla w,\nabla w)_D - 2\big|(A\nabla u,\nabla w)_D\big| - |k|^2\big((nu,u)_D + (w,w)_D + 2\big|(nu,w)_D\big|\big) \\ &\geqslant (1-\sqrt{A_+})\big((A\nabla u,\nabla u)_D + (\nabla w,\nabla w)_D\big) - |k|^2(1+\sqrt{n_+})\big((nu,u)_D + (w,w)_D\big).\end{aligned}$$

Consequently, for $k \in \mathbb{C}$ such that $|k|^2 < (A_-(1-\sqrt{A_+}))/(C_P \max(n_+,1)\ (1+\sqrt{n_+}))$, a_k^T is coercive over Y. One can then conclude thanks to the analytic Fredholm theory. □

Remark 2. In particular, if $n_+ < 1$ or if $1 < n_-$, then $\int_D (n-1) \neq 0$ and Theorem 4.4 proves that the set of interior transmission eigenvalues is discrete. Moreover, if $1 < n_-$, noticing that for $k^2 \in \mathbb{R}$,

$$\begin{aligned}\Re e\, a_k^T\big((u,w),(u,w)\big) = {} & \big(A\nabla(u-w),\nabla(u-w)\big)_D + \big((Id-A)\nabla w,\nabla w\big)_D \\ & - k^2\Big(\big(n(u-w),(u-w)\big)_D + \big((1-n)w,w\big)_D\Big),\end{aligned}$$

one can check that the first real transmission eigenvalue k_1 such that $k_1 \neq 0$ satisfies $k_1^2 \geqslant (A_- \Lambda_1(D)/n_+)$ where $\Lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D. This is one of the results of [5].

5. Case $Id < A_\star Id \leqslant A$ on a neighbourhood of ∂D

Suppose now that $Id < A_\star Id \leqslant A(x)$ a.e. on $D \cap \mathcal{V}$. Working as in the previous section with the isomorphism T defined by $T(u,w) = (u, -w + 2\chi u)$, one obtains the following theorem:

Theorem 5.1.

- *If $Id < A_\star Id \leqslant A(x)$ a.e. on $D \cap \mathcal{V}$, then $\forall k \in \mathbb{C}$, the operator $\mathcal{A}_k$ is a Fredholm operator of X.*
- *If $Id < A_\star Id \leqslant A(x)$ and $1 < n_\star \leqslant n(x)$ a.e. on $D \cap \mathcal{V}$, then the set of transmission eigenvalues is discrete in $\mathbb{C}$. Besides, there exist two positive constants ρ and δ such that if $k \in \mathbb{C}$ satisfies $|k| > \rho$ and $|\Re e\, k| < \delta|\Im m\, k|$, then k is not a transmission eigenvalue.*

With the help of the isomorphism T of Y defined by $T(u,w) = (u - \lambda(u), -w + 2u - \lambda(u))$, one can prove the following theorem:

Theorem 5.2. *Suppose $\int_D (n-1) \neq 0$ and $1 < A_-$. Then the set of transmission eigenvalues is discrete in $\mathbb{C}$. Besides, the eigenvalue of smallest magnitude k_1 such that $k_1 \neq 0$ satisfies the Faber–Krahn type estimate $|k_1|^2 \geqslant (1-1/\sqrt{A_-})/(C_P \max(n_+,1)\,(1+1/\sqrt{n_-}))$ with C_P defined in* (10)*.*

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