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Number Theory

# New results on algebraic independence with Mahler's method

## *Nouveaux résultats d'indépendance algébrique par la méthode de Mahler*

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ABSTRACT

We give some new results on algebraic independence in the frame of Mahler's method, including algebraic independence of values at transcendental points. We also give some new measures of algebraic independence. In particular, our results furnish for $n \geqslant 1$ arbitrarily large new examples of families of members $(\theta_1,\dots,\theta_n)\in\mathbb{R}^n$ normal in the sense of the definition formulated by G. Chudnovsky (1980) [2].



RÉSUMÉ

Nous donnons quelques nouveaux résultats d'indépendance algébrique de valeurs de fonctions satisfaisant des équations fonctionnelles, incluant l'indépendance algébrique en un point transcendant. Ces resultats sont obtenus avec la méthode de Mahler. En particulier, certains de nos résultats fournissent pour $n \geqslant 1$ arbitrairement grand des nouvelles familles de nombres $(\theta_1,\dots,\theta_n)\in\mathbb{R}^n$ normaux au sens de la définition formulée par G. Chudnovsky (1980) [2].



Let $p(z)=p_1(z)/p_2(z)$ be a rational function with coefficients in $\overline{\mathbb{Q}}$. Throughout this Note we set $d:=\deg p=\max(\deg p_1,\deg p_2)$, $\delta:=\mathrm{ord}_{z=0}\,p=\mathrm{ord}_{z=0}\,p_1-\mathrm{ord}_{z=0}\,p_2$.

**Definition 1.** Let $f_1(z),\dots,f_n(z)\in\overline{\mathbb{Q}}[[z]]$ be functions analytic in some neighborhood $U$ of 0, algebraically independent over $\mathbb{C}(z)$, with algebraic coefficients and satisfying a system of functional equations of the following type

$$a(z)\underline{f}(z)=A(z)\underline{f}\big(p(z)\big)+B(z), \tag{1}$$

where $\underline{f}(z)=(f_1(z),\dots,f_n(z))\in\overline{\mathbb{Q}}[[z]]^n$, $a(z)\in\overline{\mathbb{Q}}[z]$ and $A$ (resp. $B$) is an $n\times n$ (resp. $n\times 1$) matrix with coefficients in $\overline{\mathbb{Q}}[z]$. If all these conditions on $f_1(z),\dots,f_n(z)\in\overline{\mathbb{Q}}[[z]]$ are verified, we shall refer to such $n$-tuple of functions as *a set of N-functions*.

Algebraic independence of values of sets of N-functions was studied by Becker, Mahler, Nishioka, Töpfer and others [1,5–7,10]. For this purpose one can also use a general method developed in [9] (see also [8]). This method requires the so-called multiplicity estimate. Recently a new multiplicity lemma for solutions of (1) was established in [11, Theorem 3.11] and [12]. Using this multiplicity estimate with the general method from [9] one can deduce the following theorems, which

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improve certain previously known results and establish new facts on algebraic independence and measures of algebraic independence. Proofs with all the details can be found in [11].

**Theorem 2.** *Let $(f_1, \dots, f_n)$ be a set of N-functions. Assume that the rational function $p(z)$ (appearing in (1)) is a polynomial: $p(z) \in \overline{\mathbb{Q}}[z]$ and satisfies $\delta = \operatorname{ord}_{z=0} p \geqslant 2$. We also recall the notation $d = \deg p$. Let $y \in \overline{\mathbb{Q}}^*$ be such that*

$$p^{[h]}(y) \to 0$$

*(as $h \to \infty$) and no iterate $p^{[h]}(y)$ is a zero of $z \det A(z)$.*

*Then for any $\varepsilon > 0$ there is a constant $C = C(\varepsilon) > 0$ such that for any variety $W \subset \mathbb{P}^n_{\mathbb{Q}}$ of dimension $k < n + 1 - \frac{\log d}{\log \delta}$, one has*

$$\log \operatorname{Dist}(x, W) \geqslant -C\Big(h(W) + d(W)^{\frac{n+1-k+\varepsilon}{n+1-k-\frac{\log d}{\log \delta}}}\Big)^{\frac{n+1}{n-k} - \frac{\log \delta}{\log d}\frac{k+1}{n-k}} \times d(W)^{\frac{\log \delta}{\log d}\frac{k+1}{n-k}}, \tag{2}$$

*where $x = (1 : f_1(y) : \dots : f_n(y)) \in \mathbb{P}^n_{\mathbb{C}}$.*

**Remark 3.** The definition of $\operatorname{Dist}(x, W)$ (for a point $x \in \mathbb{P}^n$ and a subvariety $W$ of the same space) can be found in [4, Chapter 6, § 5], or [11, § 1.3] (see [11, Definition 1.17] and discussion after it). There are two simple cases which are important in order to understand this notion. First of all, if $W$ is a hypersurface defined by a homogeneous polynomial $P$, then $\operatorname{Dist}(x, W)$ is essentially $\|P(x)\|$ (more precisely, in this case $\log \operatorname{Dist}(x, W) = \log \|P(x)\| - \deg(P) \cdot \log \|x\| - \log \|P\|$). So essentially, for $k = n - 1$ in Theorem 2, one can substitute $\log \|P(x)\|$ in place of $\log \operatorname{Dist}(x, W)$ in the l.h.s. of (2). In this particular case one obtains an estimate which is usually known as a measure of algebraic independence.

On the other hand, for all points $x \in \mathbb{P}^n$ and all subvarieties $W \in \mathbb{P}^n$ one has $\operatorname{Dist}(x, W) = 0$ iff $x \in W$. Therefore, for $k$ as in Theorem 2 (i.e. if $k < n + 1 - \frac{\log d}{\log \delta}$) at least $k + 1$ of the values $f_1(y), \dots, f_n(y)$ are algebraically independent over $\mathbb{Q}$ (since the r.h.s. of (2) is $> -\infty$). Using this fact we readily deduce the following two corollaries:

**Corollary 4.** *Assuming the conditions of Theorem 2 one has*

$$\operatorname{tr.deg.}_{\mathbb{Q}} \mathbb{Q}\big(f_1(y), \dots, f_n(y)\big) \geqslant n + 1 - \left\lceil \frac{\log d}{\log \delta} \right\rceil.$$

**Corollary 5.** *Assuming the conditions of Theorem 2 and $\frac{\log d}{\log \delta} < 2$ one has*

$$\operatorname{tr.deg.}_{\mathbb{Q}} \mathbb{Q}\big(f_1(y), \dots, f_n(y)\big) = n. \tag{3}$$

Corollary 4 improves the lower bound

$$\operatorname{tr.deg.}_{\mathbb{Q}} \mathbb{Q}\big(f_1(y), \dots, f_n(y)\big) \geqslant \left\lceil (n+1)\frac{\log \delta}{\log d} - 1 \right\rceil,$$

established by Theorem 3 in [10], where $\lceil * \rceil$ denotes the smallest integer bigger than $*$.

Corollary 5 improves Corollary 2 of [10], where the case $n = 1$ of (3) is treated.

We can also give a measure of algebraic independence of values $y, f_1(y), \dots, f_n(y)$, for arbitrary $y \in \mathbb{C}^*$. This type of results for transcendental $y$ has not been considered before, though our estimates in this situation are weaker than in the case of algebraic $y$.

**Theorem 6.** *Let $(f_1, \dots, f_n)$ be a set of N-functions. Assume that $p(z) \in \overline{\mathbb{Q}}[z]$ with $\delta = \operatorname{ord}_{z=0} p(z) \geqslant 2$ and $d = \deg p(z)$. Let $y \in \mathbb{C}^*$ be such that*

$$p^{[h]}(y) \to 0$$

*(with $h \to \infty$) and no iterate $p^{[h]}(y)$ is a zero of $z \det A(z)$. Then for any $\varepsilon > 0$ there is a constant $C = C(\varepsilon) > 0$ such that for any variety $W \subset \mathbb{P}^{n+1}_{\mathbb{Q}}$ of dimension $k < n + 1 - 2\frac{\log d}{\log \delta}$, one has*

$$\log \operatorname{Dist}(x, W) \geqslant -C\Big(h(W) + d(W)^{\frac{n+1-k-\frac{\log d}{\log \delta}+\varepsilon}{n+1-k-2\frac{\log d}{\log \delta}}}\Big)^{2\frac{n+1}{n-k} - \frac{\log \delta}{\log d}\frac{k+1}{n-k}} \times d(W)^{\frac{\log \delta}{\log d}\frac{k+1}{n-k} - \frac{n+1}{n-k}},$$

*where $x = (1 : y : f_1(y) : \dots : f_n(y)) \in \mathbb{P}^{n+1}_{\mathbb{C}}$.*

As before, one readily deduces two corollaries:

**Corollary 7.** *Assuming the conditions of Theorem* 6 *one has*

$$\operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}\big(y, f_1(y), \dots, f_n(y)\big) \geqslant n+1-\left[2\frac{\log d}{\log\delta}\right].$$

**Corollary 8.** *Assuming the conditions of Theorem* 6 *and* $\frac{\log d}{\log\delta} < 3/2$ *one has*

$$\operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}\big(y, f_1(y), \dots, f_n(y)\big) \geqslant n-1. \tag{4}$$

The next theorem improves Theorems 1 and 2 of [10], qualitatively and quantitatively.

**Theorem 9.** *Let* $(f_1,\dots,f_n)$ *be a set of N-functions. In this statement we accept any rational function* $p(z)\in\overline{\mathbb{Q}}(z)$ *in the system* (1) (*whereas in Theorems* 2 *and* 6 *we supposed* $p(z)$ *to be a polynomial,* $p(z)\in\overline{\mathbb{Q}}[z]$). *We keep the notation* $d=\deg p$, $\delta=\operatorname{ord}_{z=0} p\geqslant 2$. *Assume that* $f_i(0)=0$, $i=1,\dots,n$. *Let* $y\in U\cap\overline{\mathbb{Q}}$ *satisfies* $\lim_{h\to\infty} p^{[h]}(y)=0$. *Assume also that for any* $h\in\mathbb{N}$ *the number* $p^{[h]}(y)$ *is not a zero of either* $\det A(z)$ *or* $z\cdot a(z)$ (*recall that* $A(z)$ *and* $a(z)$ *are introduced in Definition* 1). *Then there is a constant* $C>0$ *such that for any variety* $W\subset\mathbb{P}^n_{\mathbb{Q}}$ *of dimension* $k<2n+1-\frac{\log d}{\log\delta}(n+1)$, *one has*

$$\log\operatorname{Dist}(\underline{x},W)\geqslant -C\Big(h(W)+d(W)^{\frac{1}{1-\frac{\log d}{\log\delta}\frac{n+1}{2n-k+1}}}\Big)^{\frac{n+1}{n-k}-\frac{\log\delta}{\log d}\frac{k+1}{n-k}} d(W)^{\frac{k+1}{n-k}},$$

*where* $\underline{x}=(1:f_1(y):\ldots:f_n(y))\in\mathbb{P}^n_{\mathbb{C}}$. *In particular,*

$$\operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}\big(f_1(y), \dots, f_n(y)\big) \geqslant 2n+1-\frac{\log d}{\log\delta}(n+1). \tag{5}$$

Now we give concrete examples of sets of N-functions with $n$ elements, where $n$ can be chosen arbitrarily. In the sequel we consider a particular case of (1) when this system has the diagonal form:

$$\chi_i(z)=\chi_i\big(p(z)\big)+q_i(z),\quad i=1,\dots,n, \tag{6}$$

where $p\in\overline{\mathbb{Q}}(z)$ and $q_i\in\overline{\mathbb{Q}}[z]$, $i=1,\dots,n$. Assuming $\deg q_i\geqslant 1$ and $q_i(0)=0$, $i=1,\dots,n$, $\operatorname{ord}_{z=0} p\geqslant 2$ we obtain solutions of (6) analytic in some neighborhood of 0:

$$\chi_i(z)=\chi_i\big(p(z)\big)+q_i(z),\quad i=1,\dots,n. \tag{7}$$

Lemma 10 below allows to verify easily the algebraic independence of $\chi_1,\dots,\chi_n$ over $\mathbb{C}(z)$. It is an easy corollary of [10, Lemma 6] (as well as [3, Theorem 2]).

**Lemma 10.** *Let* $n\in\mathbb{N}^*$, $q_i\in\mathbb{C}[z]$, $i=1,\dots,n$ *and* $p\in\mathbb{C}[z]$ *satisfying* $q_i(0)=0$, $i=1,\dots,n$, $p(0)=0$ *and* $p(z)\neq z$. *Let* $\chi_1,\dots,\chi_n\in\mathbb{C}((z))$ *be functions defined by* (7). *Suppose that* $1,q_1,\dots,q_n$ *are* $\mathbb{C}$*-linearly independent and at least one of the following conditions is satisfied*:

(i) $\deg p\nmid\deg(\sum_{i=1}^n s_iq_i(z))$ *for all* $(s_1,\dots,s_n)\in\mathbb{C}^n\setminus\{\underline{0}\}$,
(ii) $\sum_{i=1}^n s_i\chi_i(z)\notin\mathbb{C}[z]$ *for all* $(s_1,\dots,s_n)\in\mathbb{C}^n\setminus\{\underline{0}\}$.

*Then the functions* $\chi_1,\dots,\chi_n$ *are algebraically independent over* $\mathbb{C}(z)$.

Using this lemma (especially point (i) which is due to Th. Töpfer) we can produce a large family of algebraically independent sets of functions (7). Obviously, all these collections $\chi_1,\dots,\chi_n$ are sets of N-functions so we can apply Theorems 2, 6, 9 and their corollaries.

**Remark 11.** In [2] G.V. Chudnovsky introduced the notion of "normality" of $n$-tuples $(x_1,\dots,x_n)\in\mathbb{C}^n$. One says that $(x_1,\dots,x_n)$ is *normal* if it has a measure of algebraic independence of the form $\exp(-Ch(P)\psi(d(P)))$, i.e. if for all polynomial $P\in\overline{\mathbb{Q}}[X_1,\dots,X_n]\setminus\{0\}$ one has the estimate

$$\big|P(x_1,\dots,x_n)\big|\geqslant\exp\big(-Ch(P)\psi\big(d(P)\big)\big), \tag{8}$$

where $C>0$ is a real constant and $\psi:\mathbb{N}\to\mathbb{R}^+$ is an arbitrary function. If one has the estimate (8) with $\psi(d)=d^\tau$ for some constant $\tau$ one says that this $n$-tuple has a measure of algebraic independence of Dirichlet's type. In this situation one also defines Dirichlet's exponent to be the infimum of $\tau$ such that $\psi(d)=d^\tau$ in (8). In [2] G.V. Chudnovsky mentioned that for $n\geqslant 2$ the examples of normal $n$-tuples are quite rare, though almost all (in the sense of Lebesgue measure) $n$-tuples of complex numbers are normal. Th. Töpfer gave a construction for a family of examples of normal $n$-tuples with Dirichlet's exponent $2n+2$ (see Theorem 1 and Corollary 4 of [10]). Our theorems assert Dirichlet's exponent $n+2$ for a subfamily of these examples and allow to produce new examples of normal $n$-tuples (due to the condition (ii) of Lemma 10).

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