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Geometry

An almost paracontact structure on a Rizza manifold

Structure presque-paracontacte sur une variété de Rizza

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ABSTRACT

We construct a class of framed $f(3,-1)$-structure on the slit tangent space of a Rizza manifold. In the special case, we show that this class induces on the indicatrix bundle an almost paracontact metric structure.

RÉSUMÉ

On construit une classe de structures repérées sur l'espace tangent marqué d'une variété de Rizza. On démontre que cette classe induit une métrique presque-paracontacte sur l'espace fibré de l'indicatrice.

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1. Introduction

In [6], G.B. Rizza introduced on almost complex manifolds (M,J), the so-called *Rizza condition*. For manifolds endowed with a Finsler metric L, the triple (M,J,L) which satisfies the Rizza condition is called *a Finsler–Rizza manifold*. It was shown by Heil [1] that if the fundamental tensor g_{ij} of the Finsler metric L is compatible with the almost complex structure J, then the Finsler structure is Riemannian. This leads to considering a weaker assumption on the Finsler metric, like the Rizza condition. The notion of Rizza manifolds was developed further in Finslerian framework by Y. Ichijyō, who showed that every tangent space to a Rizza manifold is a complex Banach space [2]. Recently, several mathematicians studied Rizza–Finsler manifolds [3–5,7].

Let (M,L) be an n-dimensional Finsler manifold (n even), admitting an almost complex structure $J^i_j(x)$. Let $g_{ij}(x,y)=\frac{1}{2}\dot\partial_i\dot\partial_j L^2(x,y)$ be the associated Finsler metric tensor field. We say that the fundamental function $L(x,y)$ satisfies *the Rizza condition*, if

$$L(x,\phi_\theta y)=L(x,y),\quad \forall\theta\in\mathbb{R},\quad \text{where } \phi^i_{\theta j}=\delta^i_j\cos\theta+J^i_j\sin\theta.$$

In this case, M is called *an almost Hermitian Finsler manifold* or simply, a *Rizza manifold*. Now let (M,J,L) be a Rizza manifold. Then

$$\widetilde{g}_{ij}(x,y)=\frac{1}{2}\big(g_{ij}(x,y)+g_{pq}(x,y)J^p_i(x)J^q_j(x)\big) \tag{1}$$

is a homogeneous symmetric generalized metric on TM, which is called a *generalized Finsler metric*; this satisfies the relation $\widetilde{g}_{pq}(x,y)J^p_i(x)J^q_j(x)=\widetilde{g}_{ij}(x,y)$. If we denote

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$$J_{ij}(x,y)=g_{im}(x,y)J^m_j(x), \qquad \widetilde{J}_{ij}(x,y)=\widetilde{g}_{im}(x,y)J^m_j(x). \tag{2}$$

Then we obtain the following relations [2,7]:

$$\widetilde{J}_{ij}=-\widetilde{J}_{ji}, \qquad \widetilde{J}_{im}J^m_j=-\widetilde{g}_{ij}, \qquad \widetilde{g}_{ij}y^j=y_i, \qquad \widetilde{g}_{ij}y^iy^j=L^2, \qquad \widetilde{J}_{ij}y^j=-J^j_iy_j, \tag{3}$$

where $y_i:=g_{ij}y^j$.

An *almost paracontact structure* on a manifold N is a set (ϕ,ξ,η), where ϕ is a tensor field of type (1, 1), ξ is a vector field and η is a 1-form such that

$$\eta(\xi)=1, \qquad \phi(\xi)=0, \qquad \eta\circ\phi=0, \qquad \phi^2=I-\eta\otimes\xi, \tag{4}$$

where I denotes the Kronecker tensor field. This structure generalizes as follows. One considers on a manifold N of dimension $(2n+s)$ a tensor field f of type $(1,1)$. If there exist on N the vector fields (ξ_a) and the 1-forms (η^a), $a=1,2,\dots,n$, such that

$$\eta^a(\xi_b)=\delta^a_b, \qquad f(\xi_a)=0, \qquad \eta^a\circ f=0, \qquad f^2=I-\sum_a\eta^a\otimes\xi_a, \tag{5}$$

then $(f,(\xi_a),(\eta^a))$ is called a framed $f(3,-1)$-structure. The term was suggested by the equation $f^3-I=0$. This is in some sense dual to the framed f-structure which generalizes the almost contact structure and which may be called a framed $f(3,1)$-structure.

Here, by using an almost complex structure $J^i_j(x)$ and the generalized Finsler metric $\widetilde{g}_{ij}(x,y)$ defined by (1) we introduce an almost product structure (G,P) on TM. Then we obtain a class of framed $f(3,-1)$-structure on $\widetilde{TM}=TM\backslash\{0\}$ and by its restriction to the indicatrix bundle IM we introduce an almost paracontact metric structure on IM.

2. A framed $f(3,-1)$-structure on $\widetilde{TM}$

Let M be a Rizza manifold, and let TM be the tangent bundle over M. We shall further use a local frame $(\delta_i,\dot{\partial}_i)$ of TM, where we put $\delta_i=\partial_i-G^m_i\dot{\partial}_m$, $\dot{\partial}_i=\frac{\partial}{\partial y^i}$, $\partial_i=\frac{\partial}{\partial x^i}$ and G^m_i are the components of an Ehresmann connection on M. Then we can globally define on TM a $(1,1)$-tensor field P, such that

$$P(\delta_i)=J^k_i\dot{\partial}_k, \qquad P(\dot{\partial}_i)=-J^k_i\delta_k. \tag{6}$$

Since $J^i_kJ^k_j=-\delta^i_j$, it is obvious that P defines an almost product structure on TM. Moreover, we can introduce an inner product $\langle\cdot,\cdot\rangle$ such that

$$\langle\delta_i,\delta_j\rangle=\widetilde{g}_{ij}, \qquad \langle\delta_i,\dot{\partial}_j\rangle=0, \qquad \langle\dot{\partial}_i,\dot{\partial}_j\rangle=\widetilde{g}_{ij}. \tag{7}$$

Then the inner product gives on TM a globally defined Riemann metric G, as follows:

$$G=\widetilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j+\widetilde{g}_{ij}\delta y^i\delta y^j, \tag{8}$$

where $(\mathrm{d}x^i,\delta y^i)$ is the dual basis of $(\delta_i,\dot{\partial}_i)$ and $\delta y^i=\mathrm{d}y^i+G^i_m\,\mathrm{d}x^m$. Using (6) and (8), we obtain

$$G\big(P(\delta_i),P(\delta_j)\big)=J^k_iJ^l_jG(\dot{\partial}_k,\dot{\partial}_l)=J^k_iJ^l_j\widetilde{g}_{kl}=\widetilde{g}_{ij}=G(\delta_i,\delta_j).$$

Similarly, we obtain $G(P(\dot{\partial}_i),P(\dot{\partial}_j))=\widetilde{g}_{ij}=G(\dot{\partial}_i,\dot{\partial}_j)$ and $G(P(\delta_i),P(\dot{\partial}_j))=0=G(\delta_i,\dot{\partial}_j)$, which ultimately lead to

Theorem 2.1. *On every Rizza manifold M, its tangent bundle TM admits an almost product structure (G,P), where P and G are defined by* (6) *and* (8)*, respectively.*

Now we define the vector fields ξ_1, ξ_2 and 1-forms η^1, η^2 on $\widetilde{TM}$ respectively by

$$\xi_1:=\alpha\big(L^2\big)y^mJ^i_m\delta_i, \qquad \xi_2:=\beta\big(L^2\big)y^i\dot{\partial}_i, \qquad \eta^1:=\gamma\big(L^2\big)y^m\widetilde{J}_{im}\,\mathrm{d}x^i, \qquad \eta^2:=\lambda\big(L^2\big)y_i\delta y^i, \tag{9}$$

where $\alpha,\beta,\gamma,\lambda:\mathbb{R}^+\to\mathbb{R}$. Using (3), (6) and (9), we obtain

$$P(\xi_1)=-\frac{\alpha}{\beta}\xi_2, \qquad P(\xi_2)=-\frac{\beta}{\alpha}\xi_1, \qquad \eta^1\circ P=-\frac{\gamma}{\lambda}\eta^2, \qquad \eta^2\circ P=-\frac{\lambda}{\gamma}\eta^1. \tag{10}$$

Using the almost product structure P, we define a new tensor field p of type $(1,1)$ on $\widetilde{TM}$ by

$$p=P-\eta^1\otimes\xi_2-\eta^2\otimes\xi_1. \tag{11}$$

By using the above relation and (10) we infer that $p(\xi_1)=-\alpha(\frac{1}{\beta}+\gamma L^2)\xi_2$, $p(\xi_2)=-\beta(\frac{1}{\alpha}+\lambda L^2)\xi_1$, $\eta^1(\xi_1)=\alpha\gamma L^2$, $\eta^2(\xi_2)=\beta\lambda L^2$ and $\eta^1(\xi_2)=\eta^2(\xi_1)=0$. Also for $X\in\mathcal{X}(\widetilde{TM})$ we get $(\eta^1\circ p)(X)=-\gamma(\frac{1}{\lambda}+\alpha L^2)\eta^2(X)$ and $(\eta^2\circ p)(X)=-\lambda(\frac{1}{\gamma}+\beta L^2)\eta^1(X)$. These relations and (10), (11) give us $p^2(X)=X+(\frac{\beta}{\alpha}+\frac{\lambda}{\gamma}+\beta\lambda L^2)\eta^1(X)\xi_1+(\frac{\alpha}{\beta}+\frac{\gamma}{\lambda}+\alpha\gamma L^2)\eta^2(X)\xi_2$. Therefore we have

Proposition 2.1. *For the triple $(p,(\xi_a),(\eta^a))$, $a=1,2$, we have*

$$p(\xi_a)=0,\qquad \eta^a\circ p=0,\qquad \eta^a(\xi_b)=\delta^a_b,\quad a,b=1,2, \tag{12}$$

$$p^2(X)=X-\eta^1(X)\xi_1-\eta^2(X)\xi_2,\quad X\in\mathcal{X}(\widetilde{TM}), \tag{13}$$

if and only if

$$\beta=-\alpha,\qquad \lambda=-\gamma,\qquad \alpha\gamma=\beta\lambda=\frac{1}{L^2}. \tag{14}$$

Theorem 2.2. *Let $p,(\xi_a),(\eta^a),a=1,2$ be defined respectively by (11) and (9). Then the triple $(p,(\xi_a),(\eta^a))$ provides a framed $f(3,-1)$-structure on $\widetilde{TM}$ if and only if (14) is hold.*

Proof. If $(p,(\xi_a),(\eta^a))$ is a framed $f(3,-1)$-structure on $\widetilde{TM}$ then from Proposition 2.1, we result that (14) is hold. Conversely, we let (14) is hold. Considering (12) and (13), in order to complete the proof, we need to prove $p^3-p=0$ and to show that p is of rank $2n-2$. Since $p(\xi_1)=p(\xi_2)=0$, by using (13) we derive that $p^3(X)=p(X)$, for all $X\in\mathcal{X}(\widetilde{TM})$. Now we need to show that $\ker p=\operatorname{span}\{\xi_1,\xi_2\}$. It is clear that $\operatorname{span}\{\xi_1,\xi_2\}\subseteq\ker p$, because $p(\xi_1)=p(\xi_2)=0$. Now let $X\in\ker p$, then $p(X)=0$ implies that $p^2(X)=0$. Thus by using (13) it follows that $X=\eta^1(X)\xi_1+\eta^2(X)\xi_2$, that is, $X\in\operatorname{span}\{\xi_1,\xi_2\}$. □

Now, we consider to the following special case of (14):

$$\alpha=1,\qquad \beta=-1,\qquad \gamma=\frac{1}{L^2},\qquad \lambda=-\frac{1}{L^2}. \tag{15}$$

Therefore, from the above theorem we result that if (15) is hold then $(p,(\xi_a),(\eta^a))$ is a framed $f(3,-1)$-structure on $\widetilde{TM}$. Note that all of the next results are hold in this spacial case. By using (11), we get the local expression of p as follows:

$$p(\delta_i)=\left(J_i^k+\frac{1}{L^2}\widetilde{J}_{ir}y^r y^k\right)\dot\partial_k,\qquad p(\dot\partial_i)=\left(-J_i^k+\frac{1}{L^2}J_r^k y^r y_i\right)\delta_k. \tag{16}$$

Relations (3), (8) and (16) give us

$$\begin{aligned} G\big(p(\delta_i),p(\delta_j)\big)&=J_i^k J_j^h\widetilde{g}_{kh}+\frac{y^r}{L^2}\left[\widetilde{J}_{jr}J_i^k y^h\widetilde{g}_{kh}+\widetilde{J}_{ir}J_j^h y^k\widetilde{g}_{kh}+\frac{1}{L^2}\widetilde{J}_{ir}\widetilde{J}_{jl}y^l y^k y^h\widetilde{g}_{kh}\right]\\ &=\widetilde{g}_{ij}-\frac{1}{L^2}\widetilde{J}_{ir}\widetilde{J}_{jk}y^r y^k. \end{aligned} \tag{17}$$

But we have $G(\delta_i,\delta_j)=\widetilde{g}_{ij}$, $\eta^1(\delta_i)\eta^1(\delta_j)=\frac{1}{L^4}y^r\widetilde{J}_{ir}y^k\widetilde{J}_{jk}$ and $\eta^2(\delta_i)\eta^2(\delta_j)=0$ which allow to rewrite (17) as $G(p(\delta_i),p(\delta_j))=G(\delta_i,\delta_j)-L^2\eta^1(\delta_i)\eta^1(\delta_j)-L^2\eta^2(\delta_i)\eta^2(\delta_j)$. We similarly obtain

$$G\big(p(\dot\partial_i),p(\dot\partial_j)\big)=\widetilde{g}_{ij}-\frac{1}{L^2}y_i y_j=G(\dot\partial_i,\dot\partial_j)-L^2\eta^1(\dot\partial_i)\eta^1(\dot\partial_j)-L^2\eta^2(\dot\partial_i)\eta^2(\dot\partial_j),$$

$$G\big(p(\delta_i),p(\dot\partial_j)\big)=0=G(\delta_i,\dot\partial_j)-L^2\eta^1(\delta_i)\eta^1(\dot\partial_j)-L^2\eta^2(\delta_i)\eta^2(\dot\partial_j).$$

Also we get $G(\delta_i,\xi_1)=G(\delta_i,y^k J_k^j\delta_j)=y^k J_k^j\widetilde{g}_{ij}=y^k\widetilde{J}_{ik}=L^2\eta^1(\delta_i)$. Similarly, we infer $G(\dot\partial_i,\xi_2)=L^2\eta^2(\dot\partial_i)$, $G(\dot\partial_i,\xi_1)=0=L^2\eta^1(\dot\partial_i)$ and $G(\delta_i,\xi_2)=0=L^2\eta^2(\delta_i)$. Therefore we conclude the following theorem:

Theorem 2.3. *The Riemannian metric G defined by (8) satisfies*

$$G\big(p(X),p(Y)\big)=G(X,Y)-L^2\eta^1(X)\eta^1(Y)-L^2\eta^2(X)\eta^2(Y),$$

$$\eta^1(X)=\frac{1}{L^2}G(X,\xi_1),\qquad \eta^2(X)=\frac{1}{L^2}G(X,\xi_2),$$

where $X,Y\in\mathcal{X}(\widetilde{TM})$.

Let us set $h(X,Y)=G(pX,Y)$ for $X,Y\in\mathcal{X}(\widetilde{TM})$. Then we have

Theorem 2.4. *The map h is a symmetric bilinear form on $\widetilde{TM}$ of rank $2n-2$, with the null space* $\operatorname{span}(\xi_1, \xi_2)$.

Proof. h is bilinear since G is so. Now, we show that h is symmetric. By using (3) and (16), we obtain $h(\delta_i, \dot{\partial}_j) = J_i^k \widetilde{g}_{kj} + \frac{1}{L^2} y^r \widetilde{J}_{ir} y^k \widetilde{g}_{kj} = \widetilde{J}_{ji} + \frac{1}{L^2} \widetilde{J}_{ir} y^r y_j$. Similarly we get $h(\dot{\partial}_i, \delta_j) = -\widetilde{J}_{ji} + \frac{1}{L^2} y_i y^r \widetilde{J}_{jr}$ and $h(\delta_i, \delta_j) = h(\dot{\partial}_i, \dot{\partial}_j) = 0$. Since $\widetilde{J}_{ij} = -\widetilde{J}_{ji}$, then from these equations we derive that $h(X, Y) = h(Y, X)$ for $X, Y \in \mathcal{X}(\widetilde{TM})$. Thus h is symmetric on $\widetilde{TM}$. Here, we prove that the null space of h is $\operatorname{span}(\xi_1, \xi_2)$. Since the null space of h is $\{X \in \chi(\widetilde{TM}) | h(X, Y) = 0,\ \forall Y \in \chi(\widetilde{TM})\} = \{X \in \chi(\widetilde{TM}) | G(pX, Y) = 0,\ \forall Y \in \chi(\widetilde{TM})\} = \{X \in \chi(\widetilde{TM}) | pX = 0\} = \ker p$, then it is sufficient that we show $\ker p = \operatorname{span}(\xi_1, \xi_2)$. By the first equation of (12) the subspace $\operatorname{span}(\xi_1, \xi_2)$ is contained in $\ker p$. Let now $X = X^i \delta_i + \dot{X}^i \dot{\partial}_i \in \ker p$, then the condition $pX = 0$ and relation (16) give $X^i = \frac{1}{L^2} X^k \widetilde{J}_{kr} y^r y^l J_l^i$ and $\dot{X}^i = \frac{1}{L^2} \dot{X}^k y_k y^i$. Hence $X = \frac{1}{L^2} X^k \widetilde{J}_{kr} y^r \xi_1 - \frac{1}{L^2} \dot{X}^k y_k \xi_2 \in \operatorname{span}(\xi_1, \xi_2)$. □

Let IM be the indicatrix bundle of (M, L), i.e., $IM = \{(x, y) \in \widetilde{TM} | L(x, y) = 1\}$, which is a submanifold of dimension $2n-1$ of $\widetilde{TM}$. Note that ξ_2 is a unit vector field on IM, since $G(\xi_2, \xi_2) = 1$. It is easy to show that ξ_2 is a normal vector field on IM with respect to the metric G. Also, ξ_2 is orthogonal to any tangent vector to IM and the vector field ξ_1 is tangent to IM, since $G(\xi_1, \xi_2) = 0$. Further, the hypersurface IM is invariant with respect to p, i.e., $p(T_u(IM)) \subseteq T_u(IM)$, $\forall u \in IM$.

Lemma 2.5. *Let the framed f-structure be given by Theorem* 2.2. *Then restricting this to IM, we have*

$$\eta^1 = y^m \widetilde{J}_{im}\, dx^i, \qquad \eta^2 = 0, \qquad p(X) = P(X) - \eta^1(X)\xi_2, \quad \forall X \in \mathcal{X}(IM).$$

Proof. Since $L^2 = 1$ on IM and $\eta^2(X) = G(X, \xi_2) = 0$, the claim follows. □

Denoting $\bar{\eta} = \eta^1|_{IM}$, $\bar{\xi} = \xi|_{IM}$, $\bar{p} = p|_{IM}$ and $\bar{G} = G|_{IM}$, then from Theorem 2.3 we get $\bar{G}(\bar{p}(X), \bar{p}(Y)) = \bar{G}(X, Y) - \bar{\eta}(X)\bar{\eta}(Y)$. Therefore Theorem 2.2 and Lemma 2.5 imply that

Theorem 2.6. *Let the framed $f(3,-1)$-structure be given by Theorem* 2.2. *Then $(\bar{p}, \bar{\xi}, \bar{\eta}, \bar{G})$ defines an almost paracontact metric structure on IM.*

References

[1] E. Heil, A relation between Finslerian and Hermitian metrics, Tensor (N.S.) 16 (1965) 1–3.
[2] Y. Ichijyō, Kählerian Finsler manifold, J. Math. Tokushima Univ. 28 (1994) 19–27.
[3] Y. Ichijyō, Almost Hermitian Finsler manifolds and nonlinear connections, in: Conf. Semin. Mat. Univ. Bari, vol. 215, 1986, pp. 1–13.
[4] N. Lee, On the special Finsler metric, Bull. Korean Math. Soc. 40 (2003) 457–464.
[5] N. Lee, D.Y. Won, Lichnerowicz connections in almost complex Finsler manifold, Bull. Korean Math. Soc. 42 (2005) 405–413.
[6] G.B. Rizza, Strutture di Finsler sulle varieta quasi complesse, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei 33 (1962) 271–275.
[7] D.Y. Won, N. Lee, Connections on almost complex Finsler manifolds and Kobayashi Hyperbolicity, J. Korean Math. Soc. 44 (2007) 237–247.