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Partial Differential Equations

A Liouville comparison principle for entire sub- and super-solutions of the equation $u_t - \Delta_p(u) = |u|^{q-1}u$

Sur un critère de comparaison de type de Liouville pour des sous- et super-solutions entières de l'équation $u_t - \Delta_p(u) = |u|^{q-1}u$

Vasilii V. Kurta

Mathematical Reviews, 416 Fourth Street, P.O. Box 8604, Ann Arbor, MI 48107-8604, USA

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ABSTRACT

We establish a Liouville comparison principle for entire sub- and super-solutions of the equation $(*)$ $w_t - \Delta_p(w) = |w|^{q-1}w$ in the half-space $\mathbb{S} = \mathbb{R}^1_+ \times \mathbb{R}^n$, where $n \geqslant 1$, $q > 0$ and $\Delta_p(w) := \mathrm{div}_x(|\nabla_x w|^{p-2}\nabla_x w)$, $1 < p \leqslant 2$. In our study we impose neither restrictions on the behaviour of entire sub- and super-solutions on the hyper-plane $t = 0$, nor any growth conditions on their behaviour or on that of any of their partial derivatives at infinity. We prove that if $1 < q \leqslant p - 1 + \frac{p}{n}$, and u and v are, respectively, an entire weak super-solution and an entire weak sub-solution of $(*)$ in $\mathbb{S}$ which belong, only locally in $\mathbb{S}$, to the corresponding Sobolev space and are such that $u \leqslant v$, then $u \equiv v$. The result is sharp. As direct corollaries we obtain both new and known Fujita-type and Liouville-type results.

RÉSUMÉ

Nous établissons un critère de comparaison de type de Liouville pour des sous- et super-solutions entières de l'équation $(*)$ $w_t - \Delta_p(w) = |w|^{q-1}w$ dans le demi-espace $\mathbb{S} = \mathbb{R}^1_+ \times \mathbb{R}^n$, où $n \geqslant 1$, $q > 0$ et $\Delta_p(w) := \mathrm{div}_x(|\nabla_x w|^{p-2}\nabla_x w)$, $1 < p \leqslant 2$. Dans notre étude, nous n'imposons ni des restrictions sur le comportement des sous- ou super-solutions entières sur le hyper-plan $t = 0$, ni des conditions de croissance sur le comportement à l'infini de ces solutions ou de leurs dérivées partielles. Nous démontrons que si $1 < q \leqslant p - 1 + \frac{p}{n}$, et u et v constituent, respectivement, une super-solution faible entière et une sous-solution faible entière de $(*)$ dans $\mathbb{S}$ qui appartiennent, localement en $\mathbb{S}$, à l'espace de Sobolev approprié, et qui sont telles que $u \leqslant v$, alors $u \equiv v$. Ce résultat est précis. Comme corollaires immédiats, nous obtenons des nouveaux résultats, ainsi que des résultats connus de type Fujita et Liouville.

1. Introduction and definitions

The purpose of this work is to obtain a Liouville comparison principle of elliptic type for entire weak sub- and super-solutions of the equation

E-mail addresses: vkurta@umich.edu, vvk@ams.org.

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$$w_t - \Delta_p(w) = |w|^{q-1}w \tag{1}$$

in the half-space $\mathbb{S} = (0, +\infty) \times \mathbb{R}^n$, where $n \geqslant 1$ is a natural number, $q > 0$ is a real number and $\Delta_p(w) := \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}x_i} A_i(\nabla w)$, with $A_i(\xi) = |\xi|^{p-2}\xi_i$ for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $p > 1$, defines the well-known p-Laplacian operator. Under entire sub- and super-solutions of (1) we understand sub- and super-solutions of (1) defined in the whole half-space $\mathbb{S}$, and under Liouville theorems of elliptic type we understand Liouville-type theorems which, in their formulations, have no restrictions on the behaviour of sub- or super-solutions to the parabolic equation (1) on the hyper-plane $t = 0$. We would also like to underline that we impose no growth conditions on the behaviour of sub- or super-solutions of (1), as well as of all their partial derivatives, at infinity.

Definition 1. Let $n \geqslant 1$, $p > 1$ and $q > 0$. A function $u = u(t, x)$ defined and measurable in $\mathbb{S}$ is called an entire weak super-solution of Eq. (1) in $\mathbb{S}$ if it belongs to the function space $L_{q,\mathrm{loc}}(\mathbb{S})$, with $u_t \in L_{1,\mathrm{loc}}(\mathbb{S})$ and $|\nabla_x u|^p \in L_{1,\mathrm{loc}}(\mathbb{S})$, and satisfies the integral inequality

$$\int_{\mathbb{S}} \left[u_t\varphi + \sum_{i=1}^{n} |\nabla_x u|^{p-2} u_{x_i}\varphi_{x_i} - |u|^{q-1}u\varphi \right] \mathrm{d}t\,\mathrm{d}x \geqslant 0 \tag{2}$$

for every non-negative function $\varphi \in C^\infty(\mathbb{S})$ with compact support in $\mathbb{S}$, where $C^\infty(\mathbb{S})$ is the space of all functions defined and infinitely differentiable in $\mathbb{S}$.

Definition 2. A function $v = v(t, x)$ is an entire weak sub-solution of (1) if $u = -v$ is an entire weak super-solution of (1).

2. Results

Theorem 1. *Let $n \geqslant 1$, $2 \geqslant p > 1$ and $1 < q \leqslant p - 1 + \frac{p}{n}$, and let u be an entire weak super-solution and v an entire weak sub-solution of (1) in $\mathbb{S}$ such that $u \geqslant v$. Then $u \equiv v$ in $\mathbb{S}$.*

The result in Theorem 1, which evidently has a comparison principle character, we term a Liouville-type comparison principle, since, in the particular cases when $u \equiv 0$ or $v \equiv 0$, it becomes a Liouville-type theorem of elliptic type, respectively, for entire sub- or super-solutions of (1).

Since in Theorem 1 we impose no conditions on the behaviour of entire sub- or super-solutions of Eq. (1) on the hyper-plane $t = 0$, we can formulate, as a direct corollary of Theorem 1, a comparison principle, which in turn one can term a Fujita comparison principle, for entire weak super- and sub-solutions u and v of the Cauchy problem, with possibly different initial data for u and v, for Eq. (1) in $\mathbb{S}$. It is clear that in the particular cases when $u \equiv 0$ or $v \equiv 0$, it becomes a Fujita-type theorem, respectively, for entire sub- or super-solutions of the Cauchy problem for Eq. (1).

Note that the result in Theorem 1 is sharp and that the hypotheses on the parameter p in Theorem 1 in fact force p to be greater than $\frac{2n}{n+1}$. The sharpness of the result for $q > p - 1 + \frac{p}{n} \geqslant 1$ follows, for example, from the existence of non-negative self-similar entire solutions to (1) in $\mathbb{S}$, which was shown in [1]. Also, there one can find a Fujita-type theorem on the non-existence of non-negative entire solutions to the Cauchy problem for (1), which was obtained as a very interesting generalisation of the famous blow-up result in [2] to quasilinear parabolic equations. For $0 < q \leqslant 1$, it is evident that the function $u(t, x) = e^t$ is a positive entire classical super-solution of (1) in $\mathbb{S}$.

We also would like to note that similar results to that in Theorem 1 for solutions of semilinear parabolic and elliptic inequalities were obtained in [3] and [4].

3. Sketch of proofs

In what follows,

$$\omega = \frac{p(q-1)}{q-p+1} \tag{3}$$

and

$$P(R) = \left\{ (t, x) \in \mathbb{S}\colon\ t^{2/\omega} + |x|^2 < R^{2/\omega} \right\}$$

for all $R > 0$. It is clear that $0 < \omega \leqslant 2$ for $1 < p \leqslant 2$ and that if $R > 0$ then

$$\text{volume of } P(R) \leqslant cR^{\frac{n+\omega}{\omega}}, \tag{4}$$

with c some positive constant which depends possibly only on n and ω.

Proof of Theorem 1. By the well-known inequality

$$\big(|u|^{q-1}u-|v|^{q-1}v\big)(u-v)\geqslant 2^{1-q}|u-v|^{q+1}$$

which holds for every $q\geqslant 1$ and all $u,v\in\mathbb{R}^1$, we obtain from (2) the relation

$$\int_{\mathbb{S}}\Bigg[(u-v)_t\varphi+\sum_{i=1}^{n}\varphi_{x_i}\big(|\nabla_x u|^{p-2}u_{x_i}-|\nabla_x v|^{p-2}v_{x_i}\big)\Bigg]\,dt\,dx\geqslant 2^{1-q}\int_{\mathbb{S}}(u-v)^q\varphi\,dt\,dx, \tag{5}$$

which holds for every non-negative function $\varphi\in C^\infty(\mathbb{S})$ with compact support in $\mathbb{S}$. Let $\tau>0$ and $R>0$ be real numbers. Let $\eta:[0,+\infty)\to[0,1]$ be a C^∞-function which has the non-negative derivative η' and equals 0 on the interval $[0,\tau]$ and 1 on the interval $[2\tau,+\infty)$, and let $\zeta:[0,+\infty)\times\mathbb{R}^n\to[0,1]$ be a C^∞-function which equals 1 on $\overline{P(R/2)}$ and 0 on $\{[0,+\infty)\times\mathbb{R}^n\}\setminus\overline{P(R)}$. Let $\varphi(t,x)=(w(t,x)+\varepsilon)^{-\nu}\zeta^s(t,x)\eta^2(t)$, where $w(t,x)=u(t,x)-v(t,x)$, $\varepsilon>0$ and the positive constants $s>1$ and $\nu\in(0,p-1)$ will be chosen below. Substituting the function φ in (5) and then integrating by parts we arrive at

$$\begin{aligned}
&-\frac{s}{1-\nu}\int_{P(R)}(w+\varepsilon)^{1-\nu}\zeta_t\zeta^{s-1}\eta^2\,dt\,dx-\frac{2}{1-\nu}\int_{P(R)}(w+\varepsilon)^{1-\nu}\zeta^s\eta'\eta\,dt\,dx\\
&\quad-\nu\int_{P(R)}\sum_{i=1}^{n}w_{x_i}\big(|\nabla_x u|^{p-2}u_{x_i}-|\nabla_x v|^{p-2}v_{x_i}\big)(w+\varepsilon)^{-\nu-1}\zeta^s\eta^2\,dt\,dx\\
&\quad+s\int_{P(R)}\sum_{i=1}^{n}\zeta_{x_i}\big(|\nabla_x u|^{p-2}u_{x_i}-|\nabla_x v|^{p-2}v_{x_i}\big)(w+\varepsilon)^{-\nu}\zeta^{s-1}\eta^2\,dt\,dx\\
&\quad\equiv I_1+I_2+I_3+I_4\geqslant 2^{1-q}\int_{P(R)}w^q(w+\varepsilon)^{-\nu}\zeta^s\eta^2\,dt\,dx.
\end{aligned}\tag{6}$$

First, observing that I_3 is non-positive, we estimate I_4 in terms of I_3 using the fact, which is a key point in our proof, that for $1<p\leqslant 2$ the p-Laplacian operator satisfies the so-called α-monotonicity condition (see, e.g., [5]) with $\alpha=p$. This in our case consists mostly of the fact that there exists a positive constant $\mathcal{K}$ such that the coefficients A_i, $i=1,\dots,n$, of the p-Laplacian operator satisfy the inequality

$$\Bigg(\sum_{i=1}^{n}\big(A_i(\xi^1)-A_i(\xi^2)\big)^2\Bigg)^{\alpha/2}\leqslant\mathcal{K}\Bigg(\sum_{i=1}^{n}\big(\xi_i^1-\xi_i^2\big)\big(A_i(\xi^1)-A_i(\xi^2)\big)\Bigg)^{\alpha-1}$$

for all pairs $\xi^1,\xi^2\in\mathbb{R}^n$ and $\alpha=p$, provided $1<p\leqslant 2$. As a result, we have

$$|I_4|\leqslant\int_{P(R)}c_1|\nabla_x\zeta|\Bigg(\sum_{i=1}^{n}w_{x_i}\big(|\nabla_x u|^{p-2}u_{x_i}-|\nabla_x v|^{p-2}v_{x_i}\big)\Bigg)^{\frac{p-1}{p}}(w+\varepsilon)^{-\nu}\zeta^{s-1}\eta^2\,dt\,dx. \tag{7}$$

Here we use the symbols c_i, $i=1,\dots,6$, to denote constants depending possibly on n, p, q, s or ν but not on R, ε or τ. Further, estimating the integrand on the right-hand side of (7) by Young's inequality we arrive at

$$\begin{aligned}
|I_4|&\leqslant\frac{\nu}{2}\int_{P(R)}\sum_{i=1}^{n}w_{x_i}\big(|\nabla_x u|^{p-2}u_{x_i}-|\nabla_x v|^{p-2}v_{x_i}\big)(w+\varepsilon)^{-\nu-1}\zeta^s\eta^2\,dt\,dx\\
&\quad+\int_{P(R)}c_2(w+\varepsilon)^{p-1-\nu}|\nabla_x\zeta|^p\zeta^{s-p}\eta^2\,dt\,dx.
\end{aligned}\tag{8}$$

Now, observing that I_2 in (6) is also non-positive, we obtain from (6) and (8) the relation

$$\begin{aligned}
&\int_{P(R)}c_2(w+\varepsilon)^{1-\nu}|\zeta_t|\zeta^{s-1}\eta^2\,dt\,dx+\int_{P(R)}c_2(w+\varepsilon)^{p-1-\nu}|\nabla_x\zeta|^p\zeta^{s-p}\eta^2\,dt\,dx\\
&\quad\geqslant\int_{P(R)}w^q(w+\varepsilon)^{-\nu}\zeta^s\eta^2\,dt\,dx+\int_{P(R)}\sum_{i=1}^{n}w_{x_i}\big(|\nabla_x u|^{p-2}u_{x_i}-|\nabla_x v|^{p-2}v_{x_i}\big)(w+\varepsilon)^{-\nu-1}\zeta^s\eta^2\,dt\,dx.
\end{aligned}\tag{9}$$

Estimating both integrands on the left-hand side of (9) by Young's inequality we obtain

$$\frac{1}{4}\int\limits_{P(R)}(w+\varepsilon)^{q-\nu}\zeta^s\eta^2\,dt\,dx + c_3\int\limits_{P(R)}|\zeta_t|^{\frac{q-\nu}{q-1}}\zeta^{s-\frac{q-\nu}{q-1}}\eta^2\,dt\,dx$$
$$+\frac{1}{4}\int\limits_{P(R)}(w+\varepsilon)^{q-\nu}\zeta^s\eta^2\,dt\,dx + c_3\int\limits_{P(R)}|\nabla_x\zeta|^{\frac{p(q-\nu)}{q-p+1}}\zeta^{s-\frac{p(q-\nu)}{q-p+1}}\eta^2\,dt\,dx$$
$$\geqslant \int\limits_{P(R)} w^q(w+\varepsilon)^{-\nu}\zeta^s\eta^2\,dt\,dx + \int\limits_{P(R)}\sum_{i=1}^n w_{x_i}\big(|\nabla_x u|^{p-2}u_{x_i} - |\nabla_x v|^{p-2}v_{x_i}\big)(w+\varepsilon)^{-\nu-1}\zeta^s\eta^2\,dt\,dx. \tag{10}$$

In (10), passing to the limit as $\varepsilon \to 0$ as justified by Lebesgue's theorem we arrive at

$$c_4\int\limits_{P(R)}|\zeta_t|^{\frac{q-\nu}{q-1}}\zeta^{s-\frac{q-\nu}{q-1}}\eta^2\,dt\,dx + c_4\int\limits_{P(R)}|\nabla_x\zeta|^{\frac{p(q-\nu)}{q-p+1}}\zeta^{s-\frac{p(q-\nu)}{q-p+1}}\eta^2\,dt\,dx \geqslant \int\limits_{P(R)} w^{q-\nu}\zeta^s\eta^2\,dt\,dx. \tag{11}$$

Now, for arbitrary $(t,x)\in\mathbb{S}$ and $R>0$, we choose in (11) the function $\zeta=\zeta(t,x)$ of the form

$$\zeta(t,x)=\psi\left(\frac{t^{2/\omega}+|x|^2}{R^{2/\omega}}\right),$$

where $0<\omega\leqslant 2$ is given by (3) and $\psi:[0,\infty)\to[0,1]$ is a C^∞-function which equals 1 on $[0,2^{-\frac{2}{\omega}}]$ and 0 on $[1,\infty)$ and is such that the inequalities

$$|\zeta_t|\leqslant c_5R^{-1} \quad\text{and}\quad |\nabla_x\zeta|\leqslant c_5R^{-\frac{1}{\omega}} \tag{12}$$

hold. Further, from (11), where we choose the parameter s sufficiently large, and (12) we obtain

$$\int\limits_{P(R/2)} w^{q-\nu}\eta^2\,dt\,dx \leqslant c_6R^{\frac{n+p}{p}-\frac{q-\nu}{q-1}}. \tag{13}$$

It is easy to calculate that for $1<q<p-1+\frac{p}{n}$ and sufficiently small ν, the inequality

$$\frac{n+p}{p}-\frac{q-\nu}{q-1}<0 \tag{14}$$

holds. Now, using (14) and passing on the right-hand side of (13) to the limit as $R\to+\infty$, we arrive at the relation

$$\int\limits_{\mathbb{S}} w^{q-\nu}\eta^2\,dt\,dx = 0$$

with $q>\nu$, which in turn, letting the parameter τ in the definition of the function η go to zero, yields that $w(t,x)=0$ a.e. in $\mathbb{S}$. Thus, we have proved Theorem 1 for $1<q<p-1+\frac{p}{n}$. Treating the case when $q=p-1+\frac{p}{n}$ requires estimating the integral

$$\int\limits_{P(R)} w^q\zeta^s\eta^2\,dt\,dx,$$

and this can be done using the relation (10) in the framework of the approach which we have used above. □

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