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Homological Algebra/Algebraic Geometry

Stratifying derived module categories

Stratification de catégories dérivées de modules

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ABSTRACT

The concept of recollement is used to obtain a stratification of the derived module category of a ring which may be regarded as an analogue of a composition series for groups or modules. This analogy raises the problem whether a 'derived' Jordan Hölder theorem holds true; that is, are such stratifications unique up to ordering and equivalence? This is indeed the case for several classes of rings, including semi-simple rings, commutative Noetherian rings, group algebras of finite groups, and finite dimensional algebras which are piecewise hereditary.

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RÉSUMÉ

On utilise la notion de recollement pour obtenir une stratification de la catégorie dérivée de la catégorie des modules sur un anneau. Ces stratifications sont des analogues des suites de composition pour les groupes et les modules. Nous sommes ainsi amenés à chercher un analogue «dérivé» du théorème de Jordan Hölder : les stratifications sont-elles uniques à l'ordre des facteurs et aux équivalences près ? C'est effectivement le cas pour plusieurs classes d'anneaux, y compris les anneaux semi-simples, les anneaux commutatifs noethériens, les algèbres de groupes de groupes finis et les algèbres de dimension finie qui sont héréditaires par morceaux.

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Les recollements de catégories triangulées ont été introduits par Beilinson, Bernstein et Deligne [5] en 1982. On peut les regarder comme suites exactes courtes au niveau des catégories triangulées, et on peut les utiliser pour déconstruire une catégorie triangulée en morceaux plus petits. Par récurrence ce procédé conduit à une «stratification» qui est un analogue d'une suite de composition, déstructurant une catégorie triangulée en strates simples. Nous sommes ainsi amenés à considérer les questions suivantes : quelles catégories triangulées sont simples, dans le sens qu'on ne peut plus les déconstruire par recollement? Y a-t-il un théorème de Jordan Hölder pour une catégorie triangulée donnée? C'est-à-dire, existe-t-il une stratification finie et les strates sont-elles uniques à l'ordre et aux équivalences près?

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Nous discutons ces questions en donnant des réponses positives pour les catégories dérivées de plusieurs classes d'anneaux. Un anneau est dit dérivé-simple si sa catégorie dérivée n'admet pas de recollement non trivial dont les termes extérieurs sont aussi des catégories dérivées d'anneaux. Notre premier résultat principal est le suivant :

Théorème 1. Les anneaux suivants sont dérivé-simples : les anneaux locaux, les anneaux artiniens simples, les anneaux commutatifs indécomposables, les blocs des algèbres de groupes de groupes finis.

Comme corollaire, le théorème de Jordan Hölder est satisfaite par les anneaux semi-simples, les anneaux commutatifs noethériens et les algèbres de groupes de groupes finis en toute caractéristique. Notre deuxième résultat principal affirme que le théorème de Jordan Hölder est vérifié pour les algèbres héréditaires par morceaux. On rappelle qu'une algèbre de dimension finie est héréditaire par morceaux s'il y a une équivalence triangulée entre sa catégorie dérivée et la catégorie dérivée d'une catégorie abélienne héréditaire.

Théorème 2. Soit *A* une algèbre héréditaire par morceaux avec n modules simples S_1, \ldots, S_n . Alors D(Mod-A) possède une stratification dont les facteurs sont les catégories dérivées des anneaux $End_A(S_i)$. Toute stratification de D(Mod-A) possède précisément ces facteurs, à l'ordre près et aux équivalences près. De plus, toute stratification de D(Mod-A) peut être réarrangée en une chaîne ascendante de catégories dérivées induite par des épimorphismes homologiques.

1. Introduction

Recollements of triangulated categories provide a tool for deconstructing the derived category of a ring A into smaller pieces. A recollement of a triangulated category \mathcal{D} is given by six functors arranged in a diagram of the form

 $\mathcal{Y} \xrightarrow{\longrightarrow} \mathcal{D} \xrightarrow{\longrightarrow} \mathcal{X}$

where \mathcal{Y} and \mathcal{X} are a triangulated subcategory and a triangulated quotient category of \mathcal{D} with the property that \mathcal{D} is obtained by glueing \mathcal{Y} and \mathcal{X} ; see below for a precise definition. We will regard recollements as analogues of short exact sequences.

In our case \mathcal{D} will either be the unbounded derived category D(Mod-A) of the category of all (right) A-modules for some ring A, or the bounded derived category $D^b(mod-A)$ of the category of finitely generated A-modules, and we will consider the following two kinds of recollements:

- (a) recollements of D(Mod-A) where the two outer terms are again unbounded derived categories of some rings B and C,
- (b) recollements of $D^b(\text{mod}-A)$ where the two outer terms are again bounded derived categories of the category of finitely generated modules over some rings *B* and *C*.

Typically, the rings *B* and *C* in the two outer terms are less complicated than *A*. For example, in a recollement of type (b) where *A*, *B*, *C* are finite dimensional algebras, the algebras *B* and *C* have smaller Grothendieck rank. One can then study *A* by investigating the two outer rings. This reduction can be employed to discuss homological properties. Indeed, since recollements induce long exact sequences for various cohomology theories, *A* and the outer terms share several homological invariants. For instance, *A* has finite global or finitistic dimension if and only if *B* and *C* have so as well, see [10,18,14,4]. A recollement of the derived category of *A* by derived categories of rings that are well understood can then be used to compute homological invariants inductively, see for example [12,13,15].

Recollements are analogues of short exact sequences, deconstructing \mathcal{D} into \mathcal{X} and \mathcal{Y} . Continuing this procedure by deconstructing \mathcal{X} and \mathcal{Y} , and so on, can be seen as stratifying \mathcal{D} by 'smaller' derived categories. If this process ends with derived categories that cannot be stratified further, these 'simple' strata can be seen as composition factors of the original derived category \mathcal{D} . At this point, there are basic questions to be asked, for any given class of rings or algebras:

- (1) Is there a finite stratification? Are then all stratifications finite?
- (2) Are the strata unique, up to ordering and derived equivalence?
- (3) Which derived categories are simple?

Questions (1) and (2) are asking for a version of the Jordan Hölder theorem for derived categories.

These questions have come up about twenty years ago in the context of viewing categories of algebraic Lie theory as highest weight categories and as module categories of quasi-hereditary algebras, see [7,14,18] for first results. Only recently, however, the technology available for triangulated and derived categories has advanced far enough to allow answers to such questions. These answers are being surveyed here. For general rings, questions (1) and (2) have negative answers; this rules out the possibility of obtaining results by purely formal arguments. Derived Jordan Hölder theorems have, however, been established for piecewise hereditary algebras, including quiver algebras and 'canonical' algebras belonging to weighted projective lines, as well as for group algebras of finite groups, in any characteristic. The latter result extends Maschke's theorem by proving that in any characteristic, group algebras are 'derived semi-simple'. The large class of 'derived simple' rings moreover includes all indecomposable commutative rings.

2. Recollements

Let \mathcal{X} , \mathcal{Y} and \mathcal{D} be triangulated categories. \mathcal{D} is said to be a *recollement* of \mathcal{X} and \mathcal{Y} if there are six triangle functors as in the following diagram

$$\mathcal{Y} \xrightarrow[i^*]{i^* = i_1 \longrightarrow j} \mathcal{D} \xrightarrow[j^i = j^* \longrightarrow j_*]{j^* \longrightarrow j^* = j^* \longrightarrow j_*} \mathcal{X} \quad \text{such that}$$

- (1) (i^*, i_*) , $(i_!, i^!)$, $(j_!, j^!)$, (j^*, j_*) are adjoint pairs;
- (2) i_* , j_* , $j_!$ are full embeddings;
- (3) $i' \circ j_* = 0$ (and thus also $j' \circ i_! = 0$ and $i^* \circ j_! = 0$);
- (4) for each $C \in D$ there are canonical triangles given by the adjunction morphisms

$$i_!i^!(C) \to C \to j_*j^*(C) \to \text{ and } j_!j^!(C) \to C \to i_*i^*(C) \to$$

Recollements were introduced by Beilinson, Bernstein and Deligne [5] in 1982. In algebraic geometry, they form a natural habitat for Grothendieck's six functors relating sheaves on a topological space with sheaves on a closed subspace and its open complement. Recollements of derived module categories occurred first around 1990 in the work of Cline, Parshall and Scott on highest weight categories [7]: the main examples were derived module categories of quasi-hereditary algebras, which admit iterated recollements by derived categories of vector spaces.

Example 2.1. *The standard recollement.* Let *A* be a ring and $e \in A$ an idempotent. According to [8], the corresponding idempotent ideal J = AeA is *stratifying*, that is, $Ae \otimes_{eAe}^{L} eA \cong AeA$, if and only if the canonical ring epimorphism $A \to A/J$ induces an embedding of derived categories $D(Mod-A/J) \hookrightarrow D(Mod-A)$. In this case we obtain a recollement

$$D(\text{Mod}-A/J) \xrightarrow{\sim} D(\text{Mod}-A) \xrightarrow{\sim} D(\text{Mod}-eAe)$$

In particular, every heredity ideal is stratifying: a two-sided ideal *J* of a finite dimensional algebra *A* is a *heredity ideal* if J = AeA for some idempotent *e* such that *eAe* is a semi-simple algebra and *J* is projective as an *A*-module. Further, *A* is *quasi-hereditary* if it admits a *heredity chain*, that is, a chain $0 = J_0 \subset J_1 \subset \cdots \subset J_s = A$ of two-sided ideals such that J_i/J_{i-1} is a heredity ideal of A/J_{i-1} for all $i \ge 1$. Then there is a sequence of iterated recollements, that is, a stratification of D(Mod-A) with strata being derived categories of simple algebras.

Schur algebras of classical groups are known to be quasi-hereditary, as well as blocks of the BGG-category O of a semisimple complex Lie algebra. In the latter case, the derived category is equivalent to a derived category of sheaves on a flag variety G/B, which itself by the Bruhat decomposition is geometrically stratified into Schubert cells BwB/B. This has motivated the definition of recollements of triangulated categories and of perverse sheaves in [5].

Example 2.2. Recollements induced by large tilting modules. Let T_C be a tilting module of projective dimension one over a ring *C* and let $A = \text{End}(T_C)$. When T_C is finitely presented, the functors $j^* = -\bigotimes_A^L T$ and $j_* = \mathbf{R} \text{Hom}_C(T, -)$ define an equivalence between D(Mod-A) and D(Mod-C) by Happel's well known result. In general, however, D(Mod-C) is only equivalent to the Verdier quotient of D(Mod-A) with respect to the kernel of the functor j^* (provided T_C is a 'good' tilting module, as shown by Bazzoni). Recently, Chen and Xi [6] have shown that j^* and j_* actually belong to a recollement

$$D(\text{Mod}-B) \xrightarrow[i^*]{i^*} D(\text{Mod}-A) \xrightarrow[j^*]{j^*} j^* D(\text{Mod}-C)$$

where the ring *B* can be computed as a universal localization in the sense of Schofield.

Examples 2.1 and 2.2 are special cases of the following construction.

Example 2.3. *Recollements induced by homological ring epimorphisms.* Let $\lambda : A \to B$ be a ring epimorphism, that is, an epimorphism in the category of rings. λ is said to be a *homological epimorphism* if $\operatorname{Tor}_i^A(B, B) = 0$ for all i > 0, or equivalently, if λ induces a fully faithful functor $i_* = \lambda_* : D(\operatorname{Mod} - B) \to D(\operatorname{Mod} - A)$. The ring epimorphism $A \to A/J$ given by a stratifying ideal in 2.1 is an example of a homological epimorphism. Also universal localization is often a homological epimorphism. By [17] every homological ring epimorphism $\lambda : A \to B$ gives rise to a recollement

$$D(\text{Mod}-B) \xrightarrow[i^*]{} \underbrace{ \xrightarrow{i^*}}_{i_*=i_! \longrightarrow} D(\text{Mod}-A) \xrightarrow[i^*]{} \underbrace{ \xrightarrow{j_!}}_{j_*} \chi$$

where $i^* = - \bigotimes_A^{\mathbf{L}} B$, $i^! = \mathbf{R} \operatorname{Hom}_A(B, -)$, and $j^* = - \bigotimes_A^{\mathbf{L}} X$ with X given by the triangle $X \to A \xrightarrow{\lambda} B \to .$ The triangulated category \mathcal{X} on the right-hand side, however, need not be a derived category of a ring.

For example, consider the Kronecker algebra A over an algebraically closed field k. The derived category D(Mod-A) has a geometric interpretation as derived category of quasi-coherent sheaves on a projective line, and it has the standard recollement from 2.1

$$D(\text{Mod}-k) \xrightarrow{\longleftarrow} D(\text{Mod}-A) \xrightarrow{\longleftarrow} D(\text{Mod}-k)$$

On the other hand, there is a homological ring epimorphism $A \rightarrow B$ where *B* is a simple Artinian ring not Morita equivalent to *k*, which is obtained as universal localization of *A* at the union **t** of all tubes in the Auslander Reiten quiver. This gives rise to a recollement

$$D(\text{Mod}-B) \xrightarrow{\longleftarrow} D(\text{Mod}-A) \xrightarrow{\longleftarrow} \mathcal{X}$$

where the right-hand side \mathcal{X} is the smallest localizing subcategory of D(Mod-A) containing **t**, see [1, Example 5.1]. Notice that \mathcal{X} is not equivalent to the derived category of a ring. Moreover, since there are no maps nor extensions between different tubes, \mathcal{X} can be decomposed further, producing an infinite stratification of D(Mod-A) by triangulated categories.

The last example shows that if we aim at a uniqueness result for stratifications of derived categories, we have to put some restriction on the triangulated categories allowed to occur in the recollements. Henceforth, we will focus on recollements of type (a) or (b) as in the introduction.

3. Derived simple rings

We say that a ring A is *derived simple* if it does not admit a non-trivial recollement of type (a), and A is $D^b(mod)$ -*derived simple* if it does not admit a non-trivial recollement of type (b).

The following result characterizes the existence of a recollement of type (a) in terms of a suitable pair of exceptional objects. Recall that $X \in D(Mod-A)$ is *exceptional* if Hom(X, X[n]) = 0 for all non-zero integers *n*. Further, *X* is *compact* if the functor Hom(X, -) preserves small coproducts, or equivalently, *X* is quasi-isomorphic to a bounded complex consisting of finitely generated projective modules. Finally, *X* is *self-compact* if Hom(X, -) preserves small coproducts inside the localizing subcategory of D(Mod-A) generated by *X*.

Theorem 3.1. (See [14,17].) There are rings A, B, C with a recollement of the form

$$D(\operatorname{Mod}-B) \xrightarrow{\longleftarrow} D(\operatorname{Mod}-A) \xrightarrow{\longleftarrow} D(\operatorname{Mod}-C)$$

if and only if there are exceptional objects $X, Y \in D(Mod-A)$ such that

- (i) X is compact, Y is self-compact, and Hom(X[n], Y) = 0 for all $n \in \mathbb{Z}$,
- (ii) an object $D \in D(Mod-A)$ is zero whenever $Hom(X \oplus Y, D[n]) = 0$ for every integer n.

In particular, $X = j_!(C)$, $Y = i_*(B)$, and $C \cong \text{End}(X)$, $B \cong \text{End}(Y)$.

Let now *A* be a local ring and $X \in D(\text{Mod}-A)$ a compact object. We can choose a representative *P* of *X* in the homotopy category $K^b(\text{proj}-A)$ of bounded complexes of finitely generated projective *A*-modules such that *P* has no direct summands of the form $X \xrightarrow{\text{Id}} X$ or its shifts for some finitely generated projective *A*-module *X*, and we can assume that *P* is of the form $\dots 0 \to P^{-n} \to \dots \to P^0 \to 0$... where P^0 has degree 0. Since P^{-n} and P^0 are actually free modules, there is a non-zero map $P^{-n} \to P^0$, which, written in matrix form, has zero entries except in one position, where the entry is an isomorphism. It gives rise to a chain map $X \to X[n]$ which is not homotopic to zero. This shows that all compact exceptional objects in D(Mod-A) are projective modules up to shift. So *A* is derived simple by Theorem 3.1, cf. [2, 4.10].

A similar argument works for simple Artinian rings. Moreover, since over a commutative ring vanishing of Hom(X, Y[n]) is determined locally, every indecomposable commutative ring is derived simple [4]. Notice, however, that the derived category D(Mod-A) may still admit recollements of triangulated categories: indeed, for commutative Noetherian rings such recollements are parametrized by the subsets of Spec A that are closed under specialization, as shown by Neeman.

There are also examples of indecomposable finite dimensional algebras with two or more non-isomorphic simple mod-

ules that are derived simple [18,9,4]. Take for instance the quiver $1 \frac{\alpha}{\beta} 2$ with relations $\alpha \beta = \beta \alpha = 0$.

A further class of examples of derived simple rings is provided by symmetric algebras. By [16] a finite dimensional indecomposable symmetric algebra over a field k is derived simple whenever it satisfies the following condition:

(\sharp) for any finitely generated non-projective *A*-module *M* there are infinitely many integers *n* with $\operatorname{Ext}_{A}^{n}(M, M) \neq 0$.

This applies for instance to group algebras of finite groups or to symmetric algebras of finite representation type. The proof uses a Calabi–Yau property of the homotopy category $K^b(\text{proj}-A)$. In general, an indecomposable triangulated category that is *d*-Calabi–Yau for some integer *d* does not admit non-trivial recollements of triangulated categories.

Theorem 3.2. The following rings are derived simple: local rings, simple Artinian rings, indecomposable commutative rings, blocks of group algebras of finite groups.

Let us now turn to $D^b(\text{mod})$ -derived simpleness. In [3,4] we carry out a detailed analysis of lifting and restriction of recollements from $D^b(\text{mod})$ to D(Mod) and vice versa when A is a finite dimensional algebra. In particular, we show that any recollement of type (b) can be lifted to a recollement of type (a). The reason is that the objects $i_*(B)$ and $j_!(C)$ in a recollement of type (b) are always compact and thus yield a pair of exceptional objects as required by Theorem 3.1. On the other hand, restriction of recollements is not always possible. However, if A has finite global dimension, then any recollement of type (a) can be restricted to a recollement of type (b). As a consequence, we obtain

Proposition 3.3. A finite dimensional algebra over a field is $D^b(mod)$ -derived simple if it is derived simple. The converse holds true provided A has finite global dimension.

Example 3.4. The radical square zero algebra given by the quiver

$$\gamma \bigcirc 1 \stackrel{\alpha}{\longleftarrow} 2 \bigcirc \beta$$

with relations $\gamma^2 = 0$, $\beta^2 = 0$ and $\alpha \circ \beta = \gamma \circ \alpha = 0$ is $D^b (\text{mod})$ -derived simple, but not derived simple. The indecomposable projective modules P_1 and P_2 and $\text{Cone}(P_2 \xrightarrow{\alpha} P_1)$ are the only indecomposable exceptional compact objects, up to shift and up to isomorphism. They give rise to non-trivial recollements of type (a) that do not restrict to $D^b (\text{mod})$ -level, see [4].

Finally, we remark that condition (\sharp) is not needed at $D^b(\text{mod})$ -level: every finite dimensional indecomposable symmetric algebra over a field k is $D^b(\text{mod})$ -derived simple [16].

4. Stratifications

By a stratification of D(Mod-A) we mean a sequence of iterated recollements



that is either infinite or ends when we reach derived simple rings at all positions. That is, a stratification is given by a binary tree with derived simple rings at the leaves.

 $D^{b}(mod)$ -stratifications are defined correspondingly by using recollements of type (b).

Question 4.1. Jordan Hölder theorem for derived categories: Given a ring A, does D(Mod-A) (or $D^b(mod-A)$) have a finite $(D^b(mod))$ -stratification which is unique up to ordering and derived equivalence of the strata?

In general, the answer will be negative. Indeed, a counterexample for the existence of finite stratifications is provided by the countable product $A = k^{\mathbb{N}}$ of a field k, see [2, 5.2]. The question of uniqueness is much more subtle. Chen and Xi exhibit in [6] rather sophisticated examples of hereditary non-Artinian rings where uniqueness fails by using the constructions described in 2.1 and 2.2. We will see below, however, that the Jordan Hölder theorem holds true for derived categories of finite dimensional hereditary algebras.

If a ring *A* has a block decomposition $A = A_1 \oplus \cdots \oplus A_s$ with derived simple blocks, then the Jordan Hölder theorem holds true. Indeed, the simple factors of any stratification are exactly the derived categories of A_1, \ldots, A_s , see [4]. Combining this with Theorem 3.2, we obtain:

Corollary 4.2. Let *A* be a semi-simple ring, or a commutative Noetherian ring, or the group algebra of a finite group. Then D(Mod-A) has a finite stratification whose factors are the derived categories of the blocks of *A*. Any stratification of D(Mod-A) has precisely these factors, up to ordering and equivalence. The corresponding result holds true for $D^b(mod)$ -stratifications.

The stratifications occurring here are just direct sum decompositions. Thus, the corollary can be restated as saying that group algebras of finite groups are derived semi-simple. This can be seen as a categorical analogue of Maschke's theorem, holding also in the modular case where the characteristic of the ground field divides the group order.

Group algebras, when not being semi-simple, have plenty of cohomology, and we have seen that the proof makes strong use of that. A very different class of algebras are path algebras of quivers; these are hereditary and there is no cohomology in degrees bigger than one. For hereditary algebras, or more generally for piecewise hereditary algebras, there are many non-trivial recollements, but the answer to Question 4.1 is still positive.

Recall that a finite dimensional algebra A over a field k is called *piecewise hereditary* if there exists a hereditary and abelian category \mathcal{H} such that the bounded derived categories $D^b(\text{mod}-A)$ and $D^b(\mathcal{H})$ are equivalent as triangulated categories. In other words, there exists a tilting complex T in $D^b(\mathcal{H})$ with endomorphism ring being A. By [11], \mathcal{H} is, up to derived equivalence, either the category mod-H for some finite dimensional hereditary k-algebra H, or the category coh(X) of coherent sheaves on an exceptional curve X (which is a weighted projective line in the sense of Geigle and Lenzing when k is algebraically closed). So, the algebra A is derived equivalent to a hereditary algebra or to a canonical algebra.

Theorem 4.3. (See [2,3].) Let A be a piecewise hereditary finite dimensional algebra over a field k, and let S_1, \ldots, S_n be representatives of the isomorphism classes of simple A-modules. Then D(Mod-A) has a stratification whose factors are the derived categories of $End_A(S_1), \ldots, End_A(S_n)$. Any stratification of D(Mod-A) has precisely these factors, up to ordering and equivalence. The corresponding result holds true for $D^b(mod)$ -stratifications.

The proof of uniqueness relies on the connection between recollements and exceptional objects. By Theorem 3.1, every recollement of type (a) comes with a compact exceptional object $X = j_1(C) \in D(Mod-A)$. When A is piecewise hereditary, there is a converse: every compact exceptional object $X \in D(Mod-A)$ induces a recollement of D(Mod-A) by the derived categories of two algebras B and C that are piecewise hereditary with at most n - 1 simple objects. More precisely, $C \cong End(X)$, and there is a homological ring epimorphism $A \rightarrow B$. As a consequence, we obtain the following stronger version of Theorem 4.3 establishing a 'normal form' for stratifications of derived categories of piecewise hereditary algebras.

Proposition 4.4. Let A be a piecewise hereditary finite dimensional algebra over a field k with n simple modules. Any stratification of D(Mod-A) can be rearranged into a chain of increasing derived module categories

$$D(Mod-A_n) \hookrightarrow \cdots \hookrightarrow D(Mod-A_2) \hookrightarrow D(Mod-A)$$

corresponding to a chain of homological epimorphisms $A = A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n$, where for each $1 \leq i < n$ there is a recollement

$$D(\text{Mod}-A_{i+1}) \xrightarrow{\longleftarrow} D(\text{Mod}-A_i) \xrightarrow{\longleftarrow} D(\text{Mod}-C_i)$$

with C_1, \ldots, C_{n-1} and A_n being derived simple.

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