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Analytic Geometry

A Note on small deformations of balanced manifolds

Une Note sur les petites déformations des variétés équilibrées

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ABSTRACT

In this Note we prove that, under a weaker condition than the $\partial\bar\partial$-lemma, the existence of balanced metrics is preserved under small deformations. This weaker condition is satisfied on the twistor space over a compact self-dual four manifold.

RÉSUMÉ

Dans cette Note nous montrons que, sous une condition plus faible que le lemme du $\partial\bar\partial$, l'existence de métriques équilibrées est préservée par des petites déformations. Cette condition affaiblie est satisfaite par l'espace des twisteurs sur une variété différentielle de dimension 4, compacte et auto-duale.

1. Introduction

Let (X, J) be a compact complex n-dimensional manifold. Let g be a hermitian metric on X and ω be the hermitian form associated to g. If $d\omega^{n-1} = 0$, then g or ω is called a *balanced* metric. A complex manifold is called a balanced manifold if it admits a balanced metric. The balanced metric was first studied thoroughly by Michelsohn [9]. She especially proved that there exists an intrinsic characterization of compact complex manifolds with balanced metrics by means of positive currents.

Theorem 1. *(See [9].) A compact complex manifold X admits a balanced metric if and only if any positive current T on X of degree $(1,1)$ must be zero if it is the component of a boundary (i.e., if there exists a current S such that $T = \partial\bar S + \bar\partial S$).*

Using this characterization, L. Alessandrini and G. Bassanelli proved the invariance of the existence of balanced metrics under modifications.

Theorem 2. *(See [2,3].) Let $f : \bar X \to X$ be a proper modification of a compact complex manifold X. Then X admits a balanced metric if and only if $\bar X$ admits a balanced metric.*

Another important question on balanced metrics involves the deformation invariance. In the following, we will assume that $\{X_t \mid t \in \triangle(\epsilon)\}$ is a complex analytic family of compact complex manifolds. Here $\triangle(\epsilon) = \{t \in \mathbb{C} \mid |t| < \epsilon\}$. In this Note,

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we only consider the small deformation, that is, we may as well assume, if necessary, ϵ will be small enough. It is well known [10] that any small deformation of a compact Kähler manifold is Kähler, that is, if X_0 is Kähler, then X_t is also Kähler for small enough t. On the other hand, the existence of a balanced metric is not preserved under small deformations. Alessandrini and Bassanelli observed in [1] that the small deformation of the Iwasawa manifold constructed by Nakamura gives such an example. However, the second author's former student C.-C. Wu in her thesis proved that in case the complex manifold satisfies the $\partial\bar{\partial}$-lemma, the balanced condition is preserved under small deformations. Actually she proved

Theorem 3. *(See [12].) If X_0 satisfies the $\partial\bar{\partial}$-lemma, then X_t also satisfies the $\partial\bar{\partial}$-lemma for small enough t.*

Theorem 4. *(See [12].) If X_0 satisfies the $\partial\bar{\partial}$-lemma and admits a balanced metric, then X_t also admits a balanced metric for sufficiently small t.*

A complex manifold satisfies the $\partial\bar{\partial}$-lemma if for its every differential form α such that $\partial\alpha = 0$ and $\bar{\partial}\alpha = 0$, and such that $\alpha = d\gamma$ for some differential form γ, there is some differential form β such that $\alpha = i\partial\bar{\partial}\beta$. There are several equivalent versions of the $\partial\bar{\partial}$-lemma, see [4]. In this Note, we consider the question of how to weaken the $\partial\bar{\partial}$-lemma condition such that the existence of balanced metrics is still preserved under small deformations. First we give

Definition 5. A compact complex n-dimensional manifold satisfies *the $(n-1,n)$-th weak $\partial\bar{\partial}$-lemma* if for its any real $(n-1, n-1)$-form φ such that $\bar{\partial}\varphi$ is a ∂-exact form, there exists an $(n-2,n-1)$-form ψ such that

$$\bar{\partial}\varphi = i\partial\bar{\partial}\psi. \tag{1}$$

Certainly the condition $(n-1,n)$-th weak $\partial\bar{\partial}$-lemma is weaker than the $\partial\bar{\partial}$-lemma. Our main result in this Note is

Theorem 6. *Let $\{X_t \mid t \in \triangle(\epsilon)\}$ be a complex analytic family of compact complex n-dimensional manifolds. Suppose X_t satisfies the $(n-1,n)$-th weak $\partial\bar{\partial}$-lemma for small $t \neq 0$. If X_0 admits a balanced metric, then for sufficiently small t, X_t also admits a balanced metric.*

So when X_t for $t \neq 0$ satisfies the $\partial\bar{\partial}$-lemma, then the existence of balanced metrics is preserved under small deformations. This result at present is mildly stronger than Theorem 4 since we don't know whether the $\partial\bar{\partial}$-lemma property is also closed under small deformations.

We should check that the small deformation of the Iwasawa manifold mentioned above does not satisfy the condition in Theorem 6. Actually if we use the notations in [1], we find that on X_t when $t \neq 0$,

$$\bar{\partial}_t(\phi_{1,t} \wedge \bar{\phi}_{1,t} \wedge \phi_{3,t} \wedge \bar{\phi}_{3,t}) = t\partial_t(\phi_{3,t} \wedge \bar{\phi}_{1,t} \wedge \bar{\phi}_{2,t} \wedge \bar{\phi}_{3,t})$$

cannot be written as a $\partial_t\bar{\partial}_t$-exact form.

An application of the main result is

Corollary 7. *Let $\{X_t \mid t \in \triangle(\epsilon)\}$ be a complex analytic family of compact complex n-dimensional manifolds. Suppose the Dolbeault cohomology group $H^{2,0}(X_t, \mathbb{C}) = 0$ for small $t \neq 0$. If X_0 admits a balanced metric, then there exists a balanced metric on X_t for sufficiently small t.*

Proof. We assume that $t \neq 0$. By Serre duality, we have $H^{n-2,n}(X_t, \mathbb{C}) = 0$. Now let φ_t be a real $(n-1,n-1)$-form on X_t such that $\bar{\partial}_t\varphi_t$ is a ∂_t-exact form, i.e., there exists an $(n-2,n)$-form η_t such that $\bar{\partial}_t\varphi_t = \partial_t\eta_t$. Since $\bar{\partial}_t\eta_t = 0$ and $H^{n-2,n}(X_t, \mathbb{C}) = 0$, there exists an $(n-2,n-1)$-form ψ_t such that $\eta_t = i\bar{\partial}_t\psi_t$. Therefore $\bar{\partial}_t\varphi_t = i\partial_t\bar{\partial}_t\psi_t$. So X_t satisfies the $(n-1,n)$-th weak $\partial\bar{\partial}$-lemma. □

Corollary 8. *Let $\{X_t \mid t \in \triangle(\epsilon)\}$ be a complex analytic family of compact complex n-dimensional manifolds. If X_0 admits a balanced metric and $H^{2,0}(X_0, \mathbb{C}) = 0$, then there exists a balanced metric on X_t for small enough t.*

Proof. Since the function $h^{p,q}(t) = \dim H^{p,q}(X_t, \mathbb{C})$ is upper semicontinuous in t, $h^{2,0}(t) \leqslant h^{2,0}(0) = 0$ for small t. So $H^{2,0}(X_t, \mathbb{C}) = 0$. Then the corollary follows. □

It is well known that the twistor space Z associated to a compact self-dual four manifold is a complex manifold and the natural metric on it is a balanced metric (cf. [9,7]). Moreover, M. Eastwood and M. Singer in [5] observed $H^{2,0}(Z, \mathbb{C}) = 0$. So above corollary implies

Corollary 9. *Let Z be the twistor space associated to a compact self-dual four manifold. Then any small deformation of Z admits a balanced metric.*

Up to now, the global deformation stability of existence of balanced metrics is unclear (cf. [13,11]). We shall use above corollary to study this question.

2. The proof of the main theorem

Let X_0 be a compact complex manifold with a balanced metric ω. Let $\pi : \mathfrak{X} \to \triangle(\epsilon)$ be a small deformation of X_0. That is, $\pi : \mathfrak{X} \to \triangle(\epsilon)$ is a holomorphic map with maximal rank so that π is proper and each fiber $X_t = \pi^{-1}(t)$ has the structure of a complex manifold which varies analytically with t (cf. [10]). Let $\phi_t : X_t \to X_0$ be a diffeomorphism which varies smoothly with t and ϕ_0 is the identity map. Since ω^{n-1} is a real d-closed $(n-1, n-1)$-form on X, $\Omega_t = \phi_t^* \omega^{n-1}$ is a real d-closed $(2n-2)$-form on X_t. We decompose Ω_t as

$$\Omega_t = \Omega_t^{n-2,n} + \Omega_t^{n-1,n-1} + \Omega_t^{n,n-2}.$$

Then we have the following facts:

(1) $\Omega_t^{n-1,n-1}$ is real and approaches ω^{n-1} as $t \to 0$;
(2) $\overline{\Omega_t^{n-2,n}} = \Omega_t^{n,n-2}$ and $\Omega_t^{n-2,n}$ approaches zero as $t \to 0$;
(3) Since Ω_t is d-closed, we have

$$\bar\partial_t \Omega_t^{n-1,n-1} + \partial_t \Omega_t^{n-2,n} = 0.$$

According to (1), $\Omega_t^{n-1,n-1}$ is strictly positive definite for sufficiently small t. Then there exists a hermitian metric ω_t on X_t such that $\omega_t^{n-1} = \Omega_t^{n-1,n-1}$. This observation is due to Michelsohn [9]. Clearly $\omega_0 = \omega$ and ω_t approaches ω smoothly and uniformly as $t \to 0$. We use this metric ω_t as the background metric on X_t.

If X_t satisfies the $(n-1, n)$-th weak $\partial\bar\partial$-lemma, then (3) implies that there exists an $(n-2, n-1)$-form Ψ_t on X_t such that

$$i\partial_t \bar\partial_t \Psi_t = \bar\partial_t \Omega_t^{n-1,n-1} = -\partial_t \Omega_t^{n-2,n}. \tag{2}$$

We can choose Ψ_t such that

$$\Psi_t \perp_{\omega_t} \ker(i\partial_t \bar\partial_t).$$

We let

$$\tilde\Omega_t = \Omega_t^{n-1,n-1} + i\partial_t \Psi_t - i\bar\partial \bar\Psi_t.$$

Then $\tilde\Omega_t$ is a d-closed real $(n-1, n-1)$-form. If we can prove $\tilde\Omega_t$ is strictly positive definite for sufficiently small t, then there exists a hermitian metric $\tilde\omega_t$ such that $\tilde\omega_t^{n-1} = \tilde\Omega_t$ and $d\tilde\omega_t^{n-1} = 0$. That is, we have gotten a balanced metric $\tilde\omega_t$ on X_t as desired.

So we only need to prove the positivity of $\tilde\Omega_t$. To this end we introduce the Kodaira–Spencer operator E_t on $\Lambda^{n-1,n}(X_t)$, i.e., on the set of all $(n-1, n)$-forms on X_t

$$E_t = \partial_t \bar\partial_t \bar\partial_t^* \partial_t^* + \partial_t^* \bar\partial_t \bar\partial_t^* \partial_t + \partial_t^* \partial_t,$$

where the Hodge-star operator $*$, which we have dropped the subscript t, is defined by the metric ω_t. We consider the equation

$$E_t(\gamma_t) = -\partial_t \Omega_t^{n-2,n} \bigl(= \bar\partial_t \Omega_t^{n-1,n-1}\bigr). \tag{3}$$

We first check that $\partial_t \Omega_t^{n-2,n} \perp_{\omega_t} \ker E_t$. In [8], Kodaira and Spencer proved that E_t is self-adjoin, strongly elliptic of order 4, and a form $\alpha_t \in \ker E_t$ if and only if

$$\partial_t \alpha_t = 0 \quad \text{and} \quad \bar\partial_t^* \partial_t^* \alpha_t = 0.$$

Then by (2), for any $\alpha_t \in \ker E_t$, we have

$$\bigl(-\partial_t \Omega_t^{n-2,n}, \alpha_t\bigr) = (i\partial_t \bar\partial_t \Psi_t, \alpha_t) = \bigl(i\Psi_t, \bar\partial_t^* \partial_t^* \alpha_t\bigr) = 0.$$

Therefore, by the theory of elliptic operators, there exists a unique smooth solution γ_t of Eq. (3) such that $\gamma_t \perp_{\omega_t} \ker E_t$.

As in [6], we can prove

$$\partial_t \gamma_t = 0 \quad \text{and} \quad i\Psi_t = \bar\partial_t^* \partial_t^* \gamma_t. \tag{4}$$

For readers' convenience, we prove these two identities here. From (2) and (3), we get $E_t(\gamma_t) - i\partial_t \bar\partial_t \Psi_t = 0$, which, from the definition of the operator E_t, is equivalent to

$$\partial_t\bar\partial_t\big(\bar\partial_t^*\partial_t^*\gamma_t - i\Psi_t\big) + \partial_t^*\big(\bar\partial_t\bar\partial_t^* + 1\big)\partial_t\gamma_t = 0.$$

By taking the L^2-norm of the left-hand side, we get

$$\partial_t\bar\partial_t\big(\bar\partial_t^*\partial_t^*\gamma_t - i\Psi_t\big) = 0 \quad \text{and} \quad \partial_t^*\big(\bar\partial_t\bar\partial_t^* + 1\big)\partial_t\gamma_t = 0. \tag{5}$$

On the other hand, for any $\phi \in \ker(i\partial_t\bar\partial_t)$, we have

$$\big(\bar\partial_t^*\partial_t^*\gamma_t, \phi\big) = (\gamma_t, \partial_t\bar\partial_t\phi) = 0.$$

Since $\Psi_t \perp_{\omega_t} \ker(i\partial_t\bar\partial_t)$,

$$\big(\bar\partial_t^*\partial_t^*\gamma_t - i\Psi_t\big) \perp_{\omega_t} \ker(i\partial_t\bar\partial_t). \tag{6}$$

Combining (5) with (6), we obtain $\bar\partial_t^*\partial_t^*\gamma_t - i\Psi_t = 0$, which is the first identity in (4). The second in (4) follows from the second equality of (5), since

$$0 = \int_{X_t} \big\langle \partial_t^*\big(\bar\partial_t\bar\partial_t^* + 1\big)\partial_t\gamma_t, \gamma_t\big\rangle = \int_{X_t} \big(\big|\bar\partial_t^*\partial_t\gamma_t\big|^2 + |\partial_t\gamma_t|^2\big).$$

Now we can estimate $\|i\partial_t\Psi_t\|_{C^0(\omega_t)}$ by elliptic estimates. Since the background metric ω_t varies smoothly with t and approaches ω_0 as $t \to 0$, we have a uniform constant C such that

$$\|i\partial_t\Psi_t\|_{C^0(\omega_t)} \leqslant C\|\gamma_t\|_{C^3(\omega_t)} \leqslant C\big\|\partial_t\Omega_t^{n-2,n}\big\|_{C^{0,\alpha}(\omega_t)} \tag{7}$$

for some $0 < \alpha < 1$. Since also $\Omega_t^{n-2,n}$ varies smoothly with t and approaches zero as $t \to 0$ and the complex structure on X_t varies analytically with t, $\|\partial_t\Omega_t^{n-2,n}\|_{C^{0,\alpha}(\omega_t)}$ approaches zero uniformly as $t \to 0$. Thus $\|i\partial_t\Psi_t\|_{C^0(\omega_t)}$ approaches zero as $t \to 0$. Therefore $\tilde\Omega_t$ is strictly positive definite when t is small enough.

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