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Factorization of linear elliptic boundary value problems in non-cylindrical domains

Factorisation de problèmes aux limites linéaires elliptiques dans un domaine non cylindrique

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ABSTRACT

We present a method of factorization for linear elliptic boundary value problems considered in non-cylindrical domains. We associate a control problem to the boundary value problem which regularizes it. The technique of change of variables is used to study this problem.

RÉSUMÉ

Nous présentons une méthode de factorisation de problèmes aux limites elliptiques linéaires dans un domaine non cylindrique. On associe au problème elliptique un problème de contrôle qui en fournit une régularisation. La technique de changement de variables est utilisée pour étudier ce problème de contrôle.

1. Introduction

In [1] Angel and Bellman proposed a method based on spatial invariant embedding for transforming a second-order elliptic boundary value problem in a rectangle, in a system of first-order decoupled initial value problems which can be solved by a two-step process. In [4], Henry and Ramos gave a complete justification for this transformation, in the case of the Poisson equation in an n-dimensional cylindrical domain. The invariant embedding was performed using the coordinate along the axis of the cylinder. The Neumann to Dirichlet (NtD) operator on a section of the cylinder was shown to satisfy a Riccati equation. The method is similar to the one used by [5], for deriving the optimal feedback for optimal control problems of parabolic equations. Its justification is based on a Galerkin method. A shorter proof was given in [2]. In this study, we generalize this method to non-cylindrical domains. The relationship with a control problem is used to regularize the Riccati equation. We use a change of variables in order to set the problem in a cylindrical domain. It was shown, in the case of the cylinder, that the LU block factorization of matrix of the problem discretized by finite differences, can be viewed as a discretization of the factorized version of the boundary value problem. Other discretizations lead to new numerical schemes.

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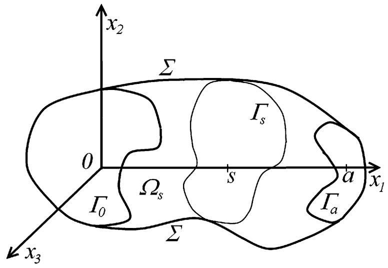

Fig. 1. The domain.

2. Elliptic boundary value problem in a non-cylindrical domain

We denote the elements of $\mathbb{R}^N$ by $(x_1, x_2, \ldots, x_N) = (x, y)$, where $x = x_1$ and $y = (x_2, \ldots, x_N)$ to stress the particular role of x_1. For each $x \in [0, a]$, let $\mathcal{O}_x$ be a bounded open set in $\mathbb{R}^{N-1}$. We shall call a quasi-cylinder with respect to x, a set Ω defined in $\mathbb{R}^N$ by $\Omega = \bigcup_{0<x<a}(x, \mathcal{O}_x)$, where $y \in \mathbb{R}^{N-1}$ is the coordinate in the section. We make the following regularity assumption on Ω: as in [3], we assume that each $\mathcal{O}_x$ has a C^2 boundary and that there exist C^2 diffeomorphisms T_x in $\mathbb{R}^{N-1}$, $T_x(\mathcal{O}_a) = \mathcal{O}_x$, where T_x denotes the flow associated with the speed field $-\frac{\partial}{\partial x} T_x(y) = V(x, T_x(y))$, continuous with respect to $x \in [0, a]$, $y \in \mathbb{R}^{N-1}$ and verifying $T_a(y) = y$. We also consider $\Sigma = \bigcup_{0<x<a}(x, \partial\mathcal{O}_x)$ to be the "lateral boundary" of the domain. Further, $\Gamma_0 = \{0\} \times \mathcal{O}_0$ and $\Gamma_a = \{a\} \times \mathcal{O}_a$ are the "faces" of the domain (see Fig. 1).

We consider the Poisson problem

$$(\mathcal{P}_0) \quad \begin{cases} -\Delta u = -\dfrac{\partial^2 u}{\partial x^2} - \Delta_y u = f, \quad \text{in } \Omega, \\ u|_\Sigma = 0, \\ -\dfrac{\partial u}{\partial x}\Big|_{\Gamma_0} = 0, \quad u|_{\Gamma_a} = u_1, \end{cases}$$

where $f \in L^2(\Omega)$ and $u_1 \in H^{1/2}_{00}(\mathcal{O}_a)$ (see [6] for the definition of this space). This problem has a unique solution in $L^2(0, a; H^2(\mathcal{O}_x) \cap H^1_0(\mathcal{O}_x)) \cap H^1(0, a; L^2(\mathcal{O}_x))$, which is the space of functions verifying $\int_0^a \|u\|^2_{H^2(\mathcal{O}_x)} \, dx < \infty$, $u_{|\partial\mathcal{O}_x} = 0$ and $\int_0^a (\|u\|^2_{L^2(\mathcal{O}_x)} + \|\frac{\partial u}{\partial x}\|^2_{L^2(\mathcal{O}_x)}) \, dx < \infty$.

As in [4], we want to factorize this problem by invariant embedding in the family of similar problems

$$(\mathcal{P}_{s,h}) \quad \begin{cases} -\Delta u_s = f \quad \text{in } \Omega_s = \displaystyle\bigcup_{0<x<s} (x, \mathcal{O}_x), \\ u_s|_{\Sigma_s} = 0, \quad -\dfrac{\partial u_s}{\partial x}\Big|_{\Gamma_0} = 0, \quad u_s|_{\Gamma_s} = h, \end{cases} \tag{1}$$

where $h \in H^{1/2}_{00}(\mathcal{O}_s)$, $\Gamma_s = \{s\} \times \mathcal{O}_s$ and $s \in]0, a[$.

By linearity of $(\mathcal{P}_0)$, for every $s \in]0, a[$ we define the Dirichlet to Neumann map through $P(s)h + w = \frac{\partial u}{\partial x}|_{\Gamma_s}$.

In Theorem 5.3 we arrive to the Riccati equation satisfied by this operator, after a change of variables.

3. Optimal control framework and regularization

We can formulate $(\mathcal{P}_0)$ as an optimal control problem:

$$(\mathcal{OC}_0) \quad \begin{cases} \dfrac{\partial u}{\partial x} = v \quad \text{in } \Omega, \quad u|_\Sigma = 0, \quad u|_{\Gamma_a} = u_1, \\ \inf\limits_v \mathcal{J}(v) = \dfrac{1}{2} \displaystyle\int_\Omega \left(|\nabla_y u - \nabla_y u_d|^2 + |v|^2\right) dx \, dy, \end{cases}$$

where u stands for the state, v for the control and u_d is given almost everywhere by

$$\begin{cases} -\Delta_y u_d(x) = f(x) \quad \text{in } \mathcal{O}_x, \\ u_d|_{\partial\mathcal{O}_x} = 0. \end{cases}$$

This is an ill-posed problem, since $\mathcal{J}$ is not defined for every $v \in L^2(\Omega)$.

In order to overcome this difficulty, we make a parabolic regularization of $(\mathcal{OC}_0)$:

$$(\mathcal{OC}_\varepsilon)\quad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial x} + \sqrt{\varepsilon}\Delta_y u_\varepsilon = v \quad \text{in } \Omega, \quad u_\varepsilon|_\Sigma = 0, \quad u_\varepsilon|_{\Gamma_a} = u_1, \\ \inf\limits_v \mathcal{J}(v) = \dfrac{1}{2}\displaystyle\int_\Omega \left(|\nabla_y u_\varepsilon - \nabla_y u_d|^2 + |v|^2\right) \mathrm{d}x\,\mathrm{d}y. \end{cases}$$

We can prove (see [5]) that this control problem has a unique solution, u_ε, in $L^2(0,a; H_0^1(\mathcal{O}_x))$.

The adjoint state p_ε is given by:

$$\begin{cases} -\dfrac{\partial p_\varepsilon}{\partial x} + \sqrt{\varepsilon}\Delta_y p_\varepsilon = -\Delta_y u_\varepsilon - f, \quad \text{in } \Omega, \\ p_\varepsilon|_\Sigma = 0, \quad p_\varepsilon|_{\Gamma_0} = 0, \end{cases}$$

and $v_\varepsilon = -p_\varepsilon$ is the optimality condition.

The problem $(\mathcal{OC}_\varepsilon)$ corresponds to the regularization of $(\mathcal{P}_0)$ with a fourth-order operator in y:

$$(\mathcal{P}_\varepsilon)\quad \begin{cases} -\dfrac{\partial^2 u_\varepsilon}{\partial x^2} - \Delta_y u_\varepsilon + \varepsilon\Delta_y^2 u_\varepsilon = f, \quad \text{in } \Omega, \\ u_\varepsilon|_\Sigma = 0, \quad -\dfrac{\partial u_\varepsilon}{\partial x}\Big|_{\Gamma_0} = 0, \quad u_{\varepsilon|\Gamma_a} = u_1, \\ \left(\dfrac{\partial u_\varepsilon}{\partial x} + \sqrt{\varepsilon}\Delta_y u_\varepsilon\right)\Big|_\Sigma = 0. \end{cases}$$

4. Change of variables

The derivation with respect to s of the invariant embedding (1), as done in [4], is not obvious as P acts on a space of functions depending on s. To avoid this problem, we are going to make a change of coordinates, following [3]. Using the flow T_x, the quasi-cylindrical domain Ω can be mapped isomorphically from the cylinder $Q =]0,a[\times\mathcal{O}_a$, by the change of variables $(x,Y)\mapsto(x,y) = (x, T_x(Y))$, $Y\in\mathcal{O}_a$. Let DT_x be the Jacobian of the transformation T_x and $J_x = \det(DT_x)$. Set $A(x)z = -J_x^{-1}\nabla_Y.(J_x(DT_x)^{-T}(DT_x)^{-1}\nabla_Y z)$ and $B(x)z = -(DT_x)^{-T}\nabla_Y z.V\circ T_x$, where $z(x,Y) = u(x,T_x(Y)) = u\circ T_x(Y)$.

Being in the framework of [3], by the change of variables the regularized problem $(\mathcal{OC}_\varepsilon)$ becomes

$$\begin{cases} \dfrac{\partial z_\varepsilon}{\partial x} - \sqrt{\varepsilon}A(x)z_\varepsilon + B(x)z_\varepsilon = v\circ T_x \quad \text{in } Q, \\ z_\varepsilon|_{\tilde\Sigma} = 0, \quad z_\varepsilon|_{\Gamma_a} = u_1, \\ \inf\limits_v \mathcal{J}(v) = \dfrac{1}{2}\displaystyle\int_Q \left(\left(DT_x^{-1}(\nabla_Y z_\varepsilon - \nabla_Y z_d)\right)^2 + |v\circ T_x|^2\right) J_x\,\mathrm{d}x\,\mathrm{d}Y, \end{cases} \tag{2}$$

with $z_d = u_d\circ T_x$ and $\tilde\Sigma =]0,a[\times\partial\mathcal{O}_a$.

Now the adjoint state is given by

$$\begin{cases} -\dfrac{\partial q_\varepsilon}{\partial x} - \sqrt{\varepsilon}A^\star(x)q_\varepsilon + B^\star(x)q_\varepsilon = A(z_\varepsilon - z_d), \quad \text{in } Q, \\ q_\varepsilon|_{\tilde\Gamma_0} = 0, \quad q_\varepsilon|_{\tilde\Sigma} = 0, \end{cases}$$

where $\tilde\Gamma_0 = \{0\}\times\mathcal{O}_a$ and the optimality condition becomes $v\circ T_x = -q_\varepsilon$. The invariant embedding with respect to the subdomains $Q_s =]0,s[\times\mathcal{O}_a$ furnishes a linear operator $\tilde P_\varepsilon(s)\in\mathcal{L}(L^2(\mathcal{O}_a), L^2(\mathcal{O}_a))$, such that the traces on $\{s\}\times\mathcal{O}_a$ satisfy $-q_\varepsilon = \tilde P_\varepsilon z_\varepsilon + r_\varepsilon$, so that z_ε verifies $\frac{\partial z_\varepsilon}{\partial x} - \sqrt{\varepsilon}A(x)z_\varepsilon + B(x)z_\varepsilon = \tilde P_\varepsilon(x)z_\varepsilon + r_\varepsilon$.

Then we can prove, as in [3], that $\tilde P_\varepsilon$ satisfies the well-posed Riccati equation in the cylinder Q, in the following sense

$$\left(\frac{\mathrm{d}\tilde P_\varepsilon}{\mathrm{d}x}\varphi,\psi\right)_{J_x} + \left(\sqrt{\varepsilon}A(x)\varphi, \tilde P_\varepsilon(x)\psi\right)_{J_x} + \left(\sqrt{\varepsilon}\tilde P_\varepsilon(x)\varphi, A(x)\psi\right)_{J_x} - \left(B(x)\varphi, \tilde P_\varepsilon(x)\psi\right)_{J_x}$$
$$- \left(\tilde P_\varepsilon(x)\varphi, B(x)\psi\right)_{J_x} + \left(\tilde P_\varepsilon(x)\varphi, \tilde P_\varepsilon(x)\psi\right)_{J_x} = \left(A(x)\varphi,\psi\right)_{J_x}, \quad \forall\varphi,\psi\in H_0^1(\mathcal{O}_a),$$

with $\tilde P_\varepsilon(0) = 0$.

5. Convergence results

We can prove the following theorem:

Theorem 5.1. *As $\varepsilon \to 0$, for $x = s$ and a fixed $z_\varepsilon(s) = h \in H_0^1(\mathcal{O}_a)$, we have $z_\varepsilon \to z$, in $H^1(Q_s)$, and $\sqrt{\varepsilon} A z_\varepsilon \to 0$, in $L^2(Q_s)$.*

Then, using (2), we can prove that

Theorem 5.2. *As $\varepsilon \to 0$, $(\frac{\partial z_\varepsilon}{\partial x} + B(x) z_\varepsilon)|_{x=s} \to (\frac{\partial z}{\partial x} + B(x) z)|_{x=s}$ in $L^2(\mathcal{O}_a)$. This implies that, for $h \in H_0^1(\mathcal{O}_a)$, $\tilde{P}_\varepsilon h \to \tilde{P} h$, in $L^2(\mathcal{O}_a)$.*

As a consequence, we obtain the desired result:

Theorem 5.3. *The Dirichlet–Neumann operator $\tilde{P}$ satisfies*

$$\begin{aligned}
&\left(\frac{\mathrm{d}\tilde{P}}{\mathrm{d}x}\varphi, \psi\right)_{J_x} - \big(B(x)\varphi, \tilde{P}(x)\psi\big)_{J_x} - \big(\tilde{P}(x)\varphi, B(x)\psi\big)_{J_x} \\
&\quad + \big(\tilde{P}(x)\varphi, \tilde{P}(x)\psi\big)_{J_x} = \big(A(x)\varphi, \psi\big)_{J_x}, \quad \forall \varphi, \psi \in H_0^1(\mathcal{O}_a).
\end{aligned} \tag{3}$$

Problem $(\mathcal{P}_0)$ is now equivalent to the factorized formulation, with $\tilde{P}$ given by (3),

$$\begin{aligned}
&\frac{\mathrm{d}r}{\mathrm{d}x} - B^\star r + \tilde{P} r = -A z_d, \qquad r(0) = 0, \\
&\frac{\mathrm{d}z}{\mathrm{d}x} + B(x) z - \tilde{P} z = r, \qquad z(a) = u_1 \quad \text{in } Q.
\end{aligned}$$

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