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Partial Differential Equations

Non-local crowd dynamics

Dynamique non locale des foules

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ABSTRACT

We present a new class of macroscopic models for pedestrian flows. Each individual is assumed to move toward a fixed target, deviating from the best path according to the crowd distribution. The resulting equation is a conservation law with a non-local flux. Each equation in this class generates a Lipschitz semigroup of solutions and is stable with respect to the functions and parameters defining it. Moreover, key qualitative properties such as the boundedness of the crowd density are proved. Two specific models in this class are considered.

RÉSUMÉ

Nous présentons ici un nouveau modèle macroscopique de trafic piéton dans lequel chaque individu se dirige vers une cible fixe en déviant du plus court chemin en fonction de la distribution de la population. On obtient une loi de conservation avec flux non local qui génère un semi-groupe de solutions et est stable par rapport aux fonctions et paramètres qu'elle contient. On montre de plus que la densité reste bornée pour tout temps. On s'intéresse plus particuliérement à deux modèles précis.

1. Introduction

From a macroscopic point of view, a moving crowd is described by its density $\rho = \rho(t,x)$. In standard situations, the number of individuals is constant, so that conservation laws of the type $\partial_t \rho + \operatorname{div}_x(\rho \mathbf{v}) = 0$ are a natural tool for the description of crowd dynamics. A key issue is the choice of the speed $\mathbf{v}$. On one hand, it describes the chosen pedestrians' path and speed. On the other hand, it also has to model the pedestrians' attitude to adapt to the crowd density they estimate to meet. Therefore, we propose the following class of Cauchy problems:

$$\begin{cases} \partial_t \rho + \operatorname{div}\big(\rho v(\rho)\big(\nu(x) + \mathcal{I}(\rho)\big)\big) = 0, \\ \rho(0,x) = \rho_o(x). \end{cases} \tag{1}$$

An individual at time t and position $x \in \mathbb{R}^N$ moves at a speed with modulus $v(\rho(t,x))$. The vector $\nu(x) + (\mathcal{I}(\rho(t)))(x)$ describes the direction that the individual located at x follows at time t, given that the density is $\rho(t)$. The vector ν is

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tangent at x to a suitable *optimal* path with respect to the visible geometry, for instance the geodesic. As soon as walls or obstacles are relevant, ν takes into consideration the discomfort felt by pedestrians, see for instance [8] and the references therein. The vector $(\mathcal{I}(\rho(t)))(x)$ describes the deviation from the direction $\nu(x)$ due to the density distribution $\rho(t)$ at time t. The operator $\mathcal{I}$ is in general *non-local*, so that $(\mathcal{I}(\rho(t)))(x)$ depends on all the values of the density $\rho(t)$ in a neighborhood of x. The constructions in [6,10,11] fit in the present setting.

Here we present two specific choices that fit in (1). A first criterion assumes that each individual aims at avoiding high crowd densities. Fix a mollifier η. Then, the convolution $(\rho * \eta)$ is an average of the crowd density around x. This leads to the natural choice, related to [1]:

$$\mathcal{I}(\rho) = -\varepsilon \frac{\nabla(\rho * \eta)}{\sqrt{1 + \|\nabla(\rho * \eta)\|^2}}, \tag{2}$$

which states that individuals deviate from the optimal path trying to avoid entering regions with higher densities. Remarkably, this model displays a pattern formation phenomenon, coherent with the widely studied feature of lane formation in pedestrian dynamics, see for instance [7,9]. Moreover, preliminary analytical investigations show that the convolution is essential in obtaining these patterns.

In (2), pedestrian evaluate the crowd density all around their position. When restrictions on the angle of vision are relevant, the following choice is reasonable:

$$\mathcal{I}(\rho) = \varepsilon \nabla \int_{\mathbb{R}^N} \rho(y)\eta(x-y)g\big((y-x)\cdot \nu(x)\big)\,\mathrm{d}y. \tag{3}$$

Here, η is as above and the smooth function g weights the deviation from the preferred direction $\nu(x)$. This choice of the operator $\mathcal{I}$ is related to [5,13].

2. Analytical results

This section is devoted to the analytical properties of (1). All proofs are deferred to [2]. In the following, $N \in \mathbb{N} \setminus \{0\}$ is the (fixed) space dimension. We denote $\mathbb{R}^+ = [0, +\infty[$; the open ball in $\mathbb{R}^N$ centered at x with radius $r > 0$ is $B(x, r)$. Let $W_N = \int_0^{\pi/2} (\cos\theta)^N \,\mathrm{d}\theta$.

Our first step in the study of (1) is the formal definition of solution.

Definition 2.1. Fix $T > 0$ and $\rho_o \in \mathbf{L}^1(\mathbb{R}^N; [0, R])$. A function $\rho \in \mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))$ is a *weak entropy solution* to (1) if it is a Kružkov solution, see [12, Definition 1], to the Cauchy problem

$$\begin{cases} \partial_t \rho + \operatorname{div}\big(\rho v(\rho) w(t,x)\big) = 0, \\ \rho(0,x) = \rho_o(x) \end{cases} \quad \text{where } w(t,x) = \nu(x) + \big(\mathcal{I}\big(\rho(t)\big)\big)(x).$$

On the functions defining the general model (1), we introduce the following hypotheses:

(**v**) $v \in \mathbf{C}^2([0, R]; [0, V])$ is non-increasing, $v(0) = V$ and $v(R) = 0$ for fixed $V, R > 0$.
($\boldsymbol{\nu}$) $\nu \in \mathbf{C}^2(\mathbb{R}^N; B(0,1))$, $\nabla\nu \in \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R}^{N\times N})$ and $\operatorname{div}\nu \in (\mathbf{W}^{1,1} \cap \mathbf{W}^{1,\infty})(\mathbb{R}^N; \mathbb{R})$.
(**I**) $\mathcal{I} \in \mathbf{C}^0(\mathbf{L}^1(\mathbb{R}^N; [0, R]); \mathbf{C}^2(\mathbb{R}^N; \mathbb{R}^N))$ satisfies the following estimates:

(**I.1**) There exists an increasing $C_I \in \mathbf{L}^\infty_{loc}(\mathbb{R}^+, \mathbb{R}^+)$ such that, for all $r \in \mathbf{L}^1(\mathbb{R}^N; [0, R])$,

$$\big\|\mathcal{I}(r)\big\|_{\mathbf{L}^\infty} + \big\|\nabla\mathcal{I}(r)\big\|_{\mathbf{L}^\infty} \leqslant C_I\big(\|r\|_{\mathbf{L}^1}\big), \qquad \big\|\mathcal{I}(r)\big\|_{\mathbf{L}^1} + \big\|\nabla\mathcal{I}(r)\big\|_{\mathbf{L}^1} \leqslant C_I\big(\|r\|_{\mathbf{L}^1}\big).$$

(**I.2**) There exists an increasing $C_I \in \mathbf{L}^\infty_{loc}(\mathbb{R}^+, \mathbb{R}^+)$ such that, for all $r \in \mathbf{L}^1(\mathbb{R}^N; [0, R])$,

$$\big\|\nabla^2\mathcal{I}(r)\big\|_{\mathbf{L}^1} \leqslant C_I\big(\|r\|_{\mathbf{L}^1}\big).$$

(**I.3**) There exists a constant K_I such that for all $r_1, r_2 \in \mathbf{L}^1(\mathbb{R}^N; [0, R])$,

$$\big\|\mathcal{I}(r_1) - \mathcal{I}(r_2)\big\|_{\mathbf{L}^\infty} + \big\|\mathcal{I}(r_1) - \mathcal{I}(r_2)\big\|_{\mathbf{L}^1} + \big\|\operatorname{div}\big(\mathcal{I}(r_1) - \mathcal{I}(r_2)\big)\big\|_{\mathbf{L}^1} \leqslant K_I \cdot \|r_1 - r_2\|_{\mathbf{L}^1}.$$

As a first justification of these conditions, we note that they make Definition 2.1 acceptable.

Lemma 2.2. *Let* (**v**), ($\boldsymbol{\nu}$) *and* **(I.1)** *hold. Choose* $r \in \mathbf{C}^0(\mathbb{R}^+; \mathbf{L}^1(\mathbb{R}^N; [0, R]))$. *Then, the Cauchy problem*

$$\begin{cases} \partial_t \rho + \operatorname{div}\big(\rho v(\rho) w(t,x)\big) = 0, \\ \rho(0,x) = \rho_o(x) \end{cases} \quad \textit{with } w(t,x) = \nu(x) + \big(\mathcal{I}\big(r(t)\big)\big)(t,x)$$

satisfies the assumptions of Kružkov theorem [12, Theorem 5].

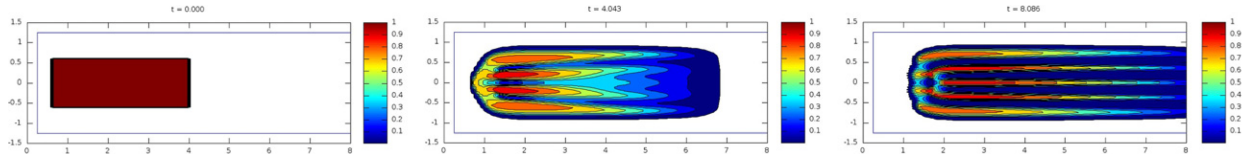

Fig. 1. Solution to (1)–(2)–(4) at time $t = 0, 4.043, 8.086$. Note the formation first of 4 and then of 5 lanes.

Fig. 1. Solution du système (1)–(2)–(4) aux temps $t = 0$, 4,043, 8,086. On remarque la formation de 4 files puis de 5.

Now, we check that the above assumption (**I**) allows us to comprehend the cases (2) and (3).

Lemma 2.3. *Fix $\varepsilon > 0$, $\eta \in \mathbf{C}_c^3(\mathbb{R}^N;\mathbb{R}^+)$ with $\int_{\mathbb{R}^N} \eta(\xi)\,\mathrm{d}\xi = 1$. Then, the operator $\mathcal{I}$ in* (2) *satisfies* (**I**)*. If moreover $g \in \mathbf{W}^{3,\infty}(\mathbb{R};[0,1])$ and $\nu \in \mathbf{W}^{3,\infty}(\mathbb{R}^N,\mathbb{R})$, then the operator $\mathcal{I}$ in* (3) *satisfies* (**I**).

We obtain a result of existence and uniqueness by iteration and use of the Banach fixed point theorem.

Theorem 2.4. *Let* (**v**), ($\boldsymbol{\nu}$) *and* (**I**) *hold. Choose any $\rho_o \in \mathbf{L}^1(\mathbb{R}^N;[0,R])$. Then, there exists a unique weak entropy solution ρ to* (1)*. Moreover, ρ satisfies the following bounds*

$$\rho(t) \in [0,R] \quad \text{for a.e. } x \in \mathbb{R}^N, \qquad \big\|\rho(t)\big\|_{\mathbf{L}^1} = \|\rho_o\|_{\mathbf{L}^1} \quad \text{for all } t \in \mathbb{R}^+,$$
$$\mathrm{TV}\big(\rho(t)\big) \leqslant \mathrm{TV}(\rho_o)e^{kt} + te^{kt} N W_N \|q\|_{\mathbf{L}^\infty([0,R])}\big(\|\nabla \operatorname{div} \nu\|_{\mathbf{L}^1} + C_I\big(\|\rho_o\|_{\mathbf{L}^1}\big)\big),$$

where we set $q(\rho) = \rho v(\rho)$ and $k = (2N+1)\|q'\|_{\mathbf{L}^\infty([0,R])}(\|\nabla \nu\|_{\mathbf{L}^\infty} + C_I(\|\rho_o\|_{\mathbf{L}^1}))$.

Using the techniques in [4,3], we obtain the continuous dependence of the solution to (1) from the initial datum and its stability with respect to v, ν and $\mathcal{I}$ in the natural norms.

Theorem 2.5. *Fix $\rho_{o,1}, \rho_{o,2} \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^N;[0,R])$. Let* (**v**), ($\boldsymbol{\nu}$) *and* (**I**) *be satisfied by*

$$\begin{cases} \partial_t \rho + \operatorname{div}\big[\rho v_1(\rho)\big(\nu_1(x) + \mathcal{I}_1(\rho)\big)\big] = 0, \\ \rho(0,x) = \rho_{o,1}(x) \end{cases} \quad \text{and} \quad \begin{cases} \partial_t \rho + \operatorname{div}\big[\rho v_2(\rho)\big(\nu_2(x) + \mathcal{I}_2(\rho)\big)\big] = 0, \\ \rho(0,x) = \rho_{o,2}(x). \end{cases}$$

Let $q_i(\rho) = \rho_i v_i(\rho_i)$. Then, the two corresponding solutions ρ_1 and ρ_2 satisfy

$$\big\|\rho_1(t) - \rho_2(t)\big\|_{\mathbf{L}^1} \leqslant C(t)\big(\|\rho_{o,1} - \rho_{o,2}\|_{\mathbf{L}^1} + \|q_1 - q_2\|_{\mathbf{W}^{1,\infty}} + \|\nu_1 - \nu_2\|_{\mathbf{L}^\infty} + \big\|\operatorname{div}(\nu_1 - \nu_2)\big\|_{\mathbf{L}^1} + d(\mathcal{I}_1, \mathcal{I}_2)\big)$$

where $d(\mathcal{I}_1,\mathcal{I}_2) = \sup\{\|\mathcal{I}_1(\rho) - \mathcal{I}_2(\rho)\|_{\mathbf{L}^\infty} + \|\operatorname{div}\mathcal{I}_1(\rho) - \operatorname{div}\mathcal{I}_2(\rho)\|_{\mathbf{L}^1}\colon \rho \in \mathbf{L}^1(\mathbb{R}^N;[0,R])\}$, the map $C \in \mathbf{C}^0(\mathbb{R}^+;\mathbb{R}^+)$ vanishes at $t = 0$ and depends on $\mathrm{TV}(\rho_{o,1})$, $\|\rho_{o,1}\|_{\mathbf{L}^1}$, $\|\nu_1\|_{\mathbf{L}^\infty}$, $\|\operatorname{div}\nu\|_{\mathbf{W}^{1,1}}$, $\|q_1\|_{\mathbf{W}^{1,\infty}}$, $\|q_2\|_{\mathbf{W}^{1,\infty}}$. (For the explicit expression of $C(t)$ we refer to [2]*).*

The above result allows us to prove the existence of optimal controls in various problems. For instance, assume that the region Ω needs to be quickly evacuated. Then, it is natural to find, for instance, the initial distribution ρ_o and the path ν such that the integral $\mathcal{J}(\rho_o,\nu) = \int_\Omega \rho(t,x)\,\mathrm{d}x$ is minimal. Theorem 2.5 ensures the continuity of $\mathcal{J}$ and, hence, the existence of minimizers in suitable compact subsets of $\mathbf{L}^1(\mathbb{R}^N;[0,R]) \times \mathbf{C}^2(\mathbb{R}^N;B(0,1))$ constrained by $\|\rho_o\|_{\mathbf{L}^1} = M$.

3. Qualitative properties

To integrate (1)–(2) we use the classical Lax–Friedrichs method with dimensional splitting. The vector $\mathcal{I}(\rho)$ needs to be computed at every time step and, due to the presence of the convolution, significantly lengthens the computation. Due to the choice of ν and of the initial data, the solution ρ vanishes in a neighborhood of 3 sides of the boundary of the computational domain. Along the exit, a free flow condition allows pedestrians to exit.

A widely detected pattern studied in crowd dynamics is that of *lane formation*. This feature has been often related to the specific qualities of each individual, i.e. it has usually been explained from a microscopic point of view. Here, in a purely macroscopic setting, we show in Fig. 1 that the solutions to (1)–(2) also display this phenomenon. Indeed, from a locally constant initial datum, first 4 lanes form and then they develop into 5 lanes. More precisely, we consider (1)–(2) with

$$\begin{aligned} &v(\rho) = \frac{1}{2}(1-\rho), \qquad \eta(x,y) = \big(1-4x^2\big)^3\big(1-4y^2\big)^3 \chi_{[-1/2,1/2]^2}(x,y), \\ &\nu(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d(x), \qquad \rho_o(x,y) = \chi_{[3/5,4]\times[-3/5,3/5]}(x), \qquad \varepsilon = 2/5, \end{aligned} \tag{4}$$

where $d = d(x)$ describes the discomfort due to walls: it is a vector normal to the walls, pointing inward, with intensity 1 along the walls, decreasing linearly to 0 at a distance 7/10 from the walls.

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