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Partial Differential Equations

Internal rectification for elastic surface waves

Rectification interne d'ondes de surface élastiques

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A R T I C L E   I N F O

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A B S T R A C T

We prove that fast oscillatory elastic surface waves can produce nontrivial internal nonoscillatory displacements.
We consider elastic surface waves of the form, in $y > 0$:

$$U^\varepsilon(t,x,y) \sim \sum_{k=2}^{\infty} \varepsilon^k U_k\left(t,x,y,\frac{x-ct}{\varepsilon},\frac{y}{\varepsilon}\right),$$

with profiles $U_k(t,x,y,Y,\theta) = \underline{U}_k(t,x,y) + U_k^*(t,x,\theta,Y)$, where U_k^* is periodic in θ and exponentially decaying to 0 in Y.
We prove that, in general, the corrector U_3 is not purely localized near the boundary, that is $\underline{U}_3$ does not vanish. U_3 depends on the slow variable y and does not decay to 0 when Y tends to $+\infty$, even if the source terms are exponentially decaying to 0.

R É S U M É

On prouve que des ondes de surface élastiques rapidement oscillantes peuvent produire un déplacement interne non oscillant non trivial.
On considère des ondes de surface élastiques de la forme, sur $y > 0$:

$$U^\varepsilon(t,x,y) \sim \sum_{k=2}^{\infty} \varepsilon^k U_k\left(t,x,y,\frac{x-ct}{\varepsilon},\frac{y}{\varepsilon}\right),$$

avec des profils $U_k(t,x,y,Y,\theta) = \underline{U}_k(t,x,y) + U_k^*(t,x,\theta,Y)$, où U_k^* est périodique en θ et exponentiellement décroissant vers 0 en Y.
On prouve que, en général, le correcteur U_3 n'est pas purement localisé près de la frontière, c'est-à-dire $\underline{U}_3$ n'est pas nul. U_3 dépend de la variable lente y et ne décroît pas vers 0 lorsque Y tend vers $+\infty$, même si les termes source sont exponentiellement décroissants vers 0.

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On prouve que des ondes de surface élastiques rapidement oscillantes peuvent produire un déplacement interne non oscillant non trivial. Ce phénomène a été observé et expliqué dans [4] pour des systèmes du premier ordre généraux ; on s'intéresse ici au cas des ondes élastiques.

On considère le problème aux limites (1)–(2) et une fréquence $(-c, 1)$ telle qu'il existe des ondes de surface oscillantes, c'est-à-dire des solutions du problème aux limites localisées près de la frontière et telles que la trace sur la frontière a des oscillations rapides en la phase $\theta = \frac{x-ct}{\varepsilon}$.

On cherche des ondes de surface qui admettent un développement asymptotique de la forme (3). U_2, le premier terme du développement, est déterminé par exemple dans [3,5] ou [6] (voir aussi [2,1,4] pour l'analyse de l'équation). Il est purement localisé près de la frontière, c'est-à-dire $\underline{U}_2 = 0$, et U_2^* est déterminé par une inconnue scalaire $\alpha_2(t, x, \theta)$, qui satisfait l'équation de propagation (4). On prouve ici que, en général, le correcteur U_3 n'est pas purement localisé près de la frontière (c'est-à-dire $\underline{U}_3 \neq 0$), même si le terme source est purement localisé près de la frontière.

$\underline{U}_3$ est solution des équations linéarisées de l'élasticité (5)–(6), avec des termes au bord déterminés par α_2 et non nuls en général. U_3 dépend de la variable lente y et ne décroît pas vers 0 lorsque Y tend vers $+\infty$, même si le terme source est exponentiellement décroissant vers 0.

1. Introduction

We prove that fast oscillatory elastic surface waves can produce nontrivial internal nonoscillatory displacements. This phenomenon was observed and explained in [4] for general first order systems. The goal of this Note is to prove that it does occur in the case of elastic waves.

For simplicity, we consider surface waves in space dimension two, on a domain which is a half-plane. We seek surface waves which admit asymptotic expansions of the form

$$U^\varepsilon(t,x,y) = \begin{pmatrix} u^\varepsilon(t,x,y) \\ v^\varepsilon(t,x,y) \end{pmatrix} \sim \sum_{k=2}^{\infty} \varepsilon^k U_k\left(t, x, y, \frac{x-ct}{\varepsilon}, \frac{y}{\varepsilon}\right),$$

on $y > 0$, with profiles $U_k(t,x,y,Y,\theta) = \underline{U}_k(t,x,y) + U_k^*(t,x,\theta,Y)$, where U_k^* is periodic in θ and exponentially decaying to 0 in Y.

The first term U_2 is determined for example in [3,5] or [6] (see also [2,1,4] for the analysis of the equation). In particular, it is purely localized near the boundary, that is $\underline{U}_2 = 0$, and U_2^* is determined by a scalar unknown $\alpha_2(t,x,\theta)$, which satisfies a propagation equation (4). Our main aim is to prove that, in general, the corrector U_3 is not purely localized near the boundary, that is $\underline{U}_3$ does not vanish, even if the source term is purely localized near the boundary. $\underline{U}_3$ is a solution of the linearized equations of elasticity, with boundary terms which are determined by α_2 and in general do not vanish. U_3 does depend on the slow variable y and does not decay to 0 as Y tends to $+\infty$, even if the source term is exponentially decaying to 0.

The other terms of the expansion can be determined as in [4]. In [4], we have proved in the case of general first order systems the existence of an exact solution admitting the given asymptotic expansion. We expect that this analysis extends to elasticity.

Following [3], we consider the problem

$$\partial_{tt}u^\varepsilon - r\partial_{xx}u^\varepsilon - (r-1)\partial_{xy}v^\varepsilon - \partial_{yy}u^\varepsilon = \partial_x F_1 + \partial_y F_2, \tag{1a}$$

$$\partial_{tt}v^\varepsilon - \partial_{xx}v^\varepsilon - (r-1)\partial_{xy}u^\varepsilon - r\partial_{yy}v^\varepsilon = \partial_x G_1 + \partial_y G_2 \quad \text{in } y > 0, \tag{1b}$$

$$\partial_y u^\varepsilon + \partial_x v^\varepsilon = -F_2 + f^\varepsilon, \tag{2a}$$

$$(r-2)\partial_x u^\varepsilon + r\partial_y v^\varepsilon = -G_2 + g^\varepsilon \quad \text{on } y = 0, \tag{2b}$$

where f^ε and g^ε are given source terms and F_1, F_2, G_1 and G_2 are quadratic terms.

We consider a particular case of the equations given in [3] (see also [5]), where the quadratic nonlinear terms are given by:

$$F_1 = \partial_y u^\varepsilon \partial_x v^\varepsilon, \qquad F_2 = \partial_x v^\varepsilon\big(\partial_x u^\varepsilon + \partial_y v^\varepsilon\big),$$
$$G_1 = \partial_y u^\varepsilon\big(\partial_x u^\varepsilon + \partial_y v^\varepsilon\big), \qquad G_2 = \partial_y u^\varepsilon \partial_x v^\varepsilon = F_1.$$

Denoting by $(-c, 1)$ one frequency such that there exist surface waves associated to the phase $\varphi(t,x) = -ct + x$ (up to a homothetic change of variable, there is no restriction in assuming that the spatial wave number $k = 1$), one looks for oscillatory surface waves, that is for solutions of the boundary value problem localized near the boundary and such that the trace on the boundary has rapid oscillations with the phase $\theta = \frac{x-ct}{\varepsilon}$.

2. Statement of the main results

2.1. Statement of the problem

Surface waves are real solutions $U^\varepsilon(t,x,y) = \binom{u^\varepsilon(t,x,y)}{v^\varepsilon(t,x,y)}$ satisfying (1) in $y>0$ and (2) on $y=0$, that admit asymptotic expansions

$$U(t,x,y) \sim \sum_{k=2}^{\infty} \varepsilon^k U_k\left(t,x,y,\frac{x-ct}{\varepsilon},\frac{y}{\varepsilon}\right), \tag{3}$$

with profiles $U_k(t,x,y,\theta,Y) = \underline{U}_k(t,x,y) + U_k^*(t,x,\theta,Y)$ belonging to the space $S = \underline{S} \oplus S^*$ defined in [4], that is U_k^* is periodic in θ and exponentially decaying in Y.

We denote by U_k^n the Fourier coefficient, with respect to θ, of order n of the profile U_k. From the definition of $S = \underline{S} \oplus S^*$, for $n \neq 0$, U_k^n is of the form $U_k^n = U_k^{n,*}(t,x,Y)$, with $U_k^{n,*} \in S^*$ and U_k^0 is of the form $U_k^0 = \underline{U}_k(t,x,y) + U_k^{0,*}(t,x,Y)$, with $\underline{U}_k \in \underline{S}$ and $U_k^{0,*} \in S^*$.

For the sake of definitiveness, we suppose that the solution vanishes identically in the past: $\forall t \leqslant 0$, $U^\varepsilon(t) = 0$, and, to fix the ideas, that it is ignited by source terms f^ε and g^ε on the boundary, which we assume to be small and localized near the boundary:

$$f(t,x) \sim \sum_{k=2}^{\infty} \varepsilon^k f_k\left(t,x,\frac{x-ct}{\varepsilon}\right), \qquad g(t,x) \sim \sum_{k=2}^{\infty} \varepsilon^k g_k\left(t,x,\frac{x-ct}{\varepsilon}\right),$$

with profiles f_k and g_k belonging to S^*, that is exponentially decaying to 0 as Y tends to $+\infty$, and vanishing identically in the past: $\forall t \leqslant 0$, $f^\varepsilon(t) = g^\varepsilon(t) = 0$.

Remark 2.1. The order of magnitude $U = O(\varepsilon^2)$ and $f, g = O(\varepsilon^2)$ as $\varepsilon \to 0$ corresponds to the regime of weakly nonlinear geometric optics where the nonlinear effects are present in the propagation of the leading term U_2.

2.2. Main results

Theorem 2.2. *The profile of main order U_2 belongs to S^*, i.e. is purely localized on the boundary: $\underline{U}_2$ vanishes.*

U_2^* is determined by a scalar unknown $\alpha_2(t,x,\theta) = \sum_{n\in\mathbb{Z}^*} \alpha_2(n,t,x)e^{in\theta}$ (see Section 4 for an explicit form of U_2^*), which solves an equation

$$\partial_t\alpha_2 + c\partial_x\alpha_2 + a(\alpha_2,\alpha_2) = l(f_2,g_2) \tag{4}$$

where $l(f_2,g_2)$, up to a multiplicative constant, has Fourier coefficients of order n equal to $\frac{p_1}{|n|} f_2^n + \frac{q}{in} g_2^n$ (p_1 and q given by (11)) and a is a nonlocal bilinear form such that the Fourier coefficients of order n of $a(\alpha_2,\alpha_2)$ are equal to

$$\sum_{k'=1}^{k} \Lambda_1(k,k')k'(k-k')\alpha_2(k')\alpha_2(k-k') + \sum_{k'=k}^{\infty} \Lambda_2(k,k')(k-k')\alpha_2(k')\alpha_2(k-k'),$$

where the expressions of the kernels Λ_1 and Λ_2 are given in [3].

This theorem is proved in Section 6: we show that $\underline{U}_2$ solves the linearized equations of elasticity with homogeneous boundary conditions, so that it must vanish. The relation between U_2^* and α_2 as well as the equation for α_2 are obtained in [3].

The main results of the Note are the following theorem and corollary:

Theorem 2.3. *The profile of higher order U_3 is of the form $U_3 = \underline{U}_3 + U_3^*$, with $U_3^* = \binom{u_3^*}{v_3^*} \in S^*$ and $\underline{U}_3 = \binom{\underline{u}_3}{\underline{v}_3} \in \underline{S}$ satisfying*

$$\partial_{tt}\underline{u}_3 - r\partial_{xx}\underline{u}_3 - (r-1)\partial_{xy}\underline{v}_3 - \partial_{yy}\underline{u}_3 = 0, \tag{5a}$$

$$\partial_{tt}\underline{v}_3 - \partial_{xx}\underline{v}_3 - (r-1)\partial_{xy}\underline{u}_3 - r\partial_{yy}\underline{v}_3 = 0 \quad \text{in } y>0, \tag{5b}$$

$$\partial_y\underline{u}_3 + \partial_x\underline{v}_3 = A_r \sum_{n\in Z^*} |n|\partial_x\big(\big|\alpha_2(n,t,x)\big|\big)^2, \quad \text{with } A_r = -\frac{2}{r}\frac{q}{p_1+p_2}C_r \neq 0, \tag{6a}$$

$$(r-2)\partial_x\underline{u}_3 + r\partial_y\underline{v}_3 = 0 \quad \text{on } y=0. \tag{6b}$$

These equations are obtained in Section 7.

Remark 2.4. U_3^* is determined by U_2 and a scalar unknown $\alpha_3(t,x,\theta)$, which solves the linearized equation of (4).

Corollary 2.5. *If f_2 and g_2 satisfy $\partial_x l(f_2,g_2)|_{t=0} \neq 0$, then $\underline{U}_3 \neq 0$.*

Proof. Since u_2 and v_2, and thus α_2, vanish in the past, we have $\partial_t\alpha_2|_{t=0} = l(f_2,g_2)|_{t=0}$, and therefore, for t small, $\alpha_2 \sim tl(f_2,g_2)$; we then obtain that $\partial_x l(f_2,g_2)|_{t=0} \neq 0$ yields $\sum_{n\in Z^*} |n|\partial_x(|\alpha_2(n,t,x)|)^2 \neq 0$.

The right-hand side term of the first boundary condition satisfied by $\underline{U}_3$ does not vanish, we then obtain that $\underline{U}_3$ does not vanish. □

Remark 2.6. For example, we can take f_1 and f_2 such that their Fourier coefficients of order 1 satisfy $\partial_x(p_1 f_2^1 - iqg_2^1)|_{t=0} \neq 0$ in order to have $\partial_x l(f_2,g_2)|_{t=0} \neq 0$ and thus $\underline{U}_3 \neq 0$.

3. The cascade of equations

Plugging the expression (3) of U^ε into the equations and boundary conditions and collecting the powers of ε yield, for all $k \geqslant 2$,

$$\left(c^2 - r\right)\partial_{\theta\theta}u_k - (r-1)\partial_{\theta Y}v_k - \partial_{YY}u_k = H_{k-1}(u_{k-1}, v_{k-1}, \ldots, u_2, v_2), \tag{7a}$$

$$\left(c^2 - 1\right)\partial_{\theta\theta}v_k - (r-1)\partial_{\theta Y}u_k - r\partial_{YY}v_k = K_{k-1}(u_{k-1}, v_{k-1}, \ldots, u_2, v_2) \quad \text{on } \{Y > 0,\ y > 0\} \tag{7b}$$

and

$$\partial_Y u_k + \partial_\theta v_k = h_{k-1}(u_2, v_2, \ldots, u_{k-1}, v_{k-1}), \tag{8a}$$

$$(r-2)\partial_\theta u_k + r\partial_Y v_k = k_{k-1}(u_2, v_2, \ldots, u_{k-1}, v_{k-1}) \quad \text{on } Y = y = 0. \tag{8b}$$

In particular, $H_1 = K_1 = 0$, $h_1 = k_1 = 0$ and H_3 and h_3 are given by (the expressions of H_k, K_k, h_k and k_k for general k being similar):

$$\begin{aligned} H_3 = {} & 2c\partial_{\theta t}u_3 + 2r\partial_{\theta x}u_3 + (r-1)\partial_{\theta y}v_3 + (r-1)\partial_{Yx}v_3 + 2\partial_{Yy}u_3 - \partial_{tt}u_2 + r\partial_{xx}u_2 + (r-1)\partial_{xy}v_2 + \partial_{yy}u_2 \\ & + \partial_x[\partial_Y u_2 \partial_\theta v_2] + \partial_y\left[\partial_\theta v_2(\partial_\theta u_2 + \partial_Y v_2)\right] + \partial_Y\left[\partial_\theta v_2(\partial_\theta u_3 + \partial_Y v_3 + \partial_x u_2 + \partial_y v_2)\right. \\ & \left. + (\partial_\theta v_3 + \partial_x v_2)(\partial_\theta u_2 + \partial_Y v_2)\right] + \partial_\theta\left[\partial_Y u_2(\partial_\theta v_3 + \partial_x v_2) + \partial_\theta v_2(\partial_Y u_3 + \partial_y u_2)\right], \end{aligned} \tag{9}$$

$$h_3 = -\partial_y u_3 - \partial_x v_3 - \partial_\theta v_2(\partial_\theta u_3 + \partial_Y v_3 + \partial_x u_2 + \partial_y v_2) - (\partial_\theta v_3 + \partial_x v_2)(\partial_\theta u_2 + \partial_Y v_2) + f_3. \tag{10}$$

4. Form of the oscillatory parts of the profile U_2

We define p_1, p_2 and q by

$$p_1^2 = 1 - c^2, \quad p_1 < 0, \qquad p_2^2 = 1 - \frac{c^2}{r}, \quad p_2 < 0, \qquad q^2 = p_1 p_2, \quad q > 0. \tag{11}$$

In order to have the existence of surface waves, the following assumption has to be satisfied:

Assumption 4.1. *We assume that*

$$\left(2 - c^2\right)^2 = 4q^2. \tag{12}$$

We obtain the following form of the Fourier coefficients u_2^n, v_2^n, $n \neq 0$

$$u_2^n(t,x,Y) = q\alpha_2(n,t,x)\left(qe^{p_1|n|Y} - e^{p_2|n|Y}\right), \tag{13}$$

$$v_2^n(t,x,Y) = i\,\mathrm{sign}(n)p_2\alpha_2(n,t,x)\left(-e^{p_1|n|Y} + qe^{p_2|n|Y}\right), \tag{14}$$

where α_2 is a scalar unknown satisfying Eq. (4).

5. Equation and boundary condition for U_k^0

5.1. Equations

The profiles u_k^0 and v_k^0 satisfy

$$-\partial_{YY} u_k^0 = H_{k-1}^0, \qquad -r\partial_{YY} v_k^0 = K_{k-1}^0 \quad \text{in } y > 0, Y > 0. \tag{15}$$

Since $u_k^0, v_k^0 \in S$, we obtain the resolubility conditions

$$\underline{H}_{k-1}^0 = \underline{K}_{k-1}^0 = 0. \tag{16}$$

Eqs. (15) yield

$$u_k^0 = \underline{u}_k^0(t,x,y) + u_k^{0,*}(t,x,y,Y), \qquad v_k^0 = \underline{v}_k^0(t,x,y) + v_k^{0,*}(t,x,y,Y), \tag{17}$$

with $\underline{u}_k^0, \underline{v}_k^0 \in \underline{S}$ unknown functions that have to be determined and $u_k^{0,*}, v_k^{0,*} \in S^*$ known functions given by the following expressions:

$$u_k^{0,*} = -\int_Y^\infty \left(\int_s^\infty H_{k-1}^0(t,x,y,s')\,\mathrm{d}s' \right) \mathrm{d}s, \tag{18}$$

$$v_k^{0,*} = -\frac{1}{r} \int_Y^\infty \left(\int_s^\infty K_{k-1}^0(t,x,y,s')\,\mathrm{d}s' \right) \mathrm{d}s. \tag{19}$$

5.2. Boundary conditions for Fourier coefficients u_k^0 and v_k^0

The boundary conditions for u_k^0 and v_k^0 read

$$\partial_Y u_k^0 = h_{k-1}^0, \qquad r\partial_Y v_k^0 = k_{k-1}^0, \quad \text{on } Y = y = 0. \tag{20}$$

Plugging the expressions (17) in (20), we obtain

$$\int_0^\infty H_{k-1}^0(t,x,y,s)\,\mathrm{d}s = h_{k-1}^0(t,x,y,Y), \tag{21a}$$

$$\int_0^\infty K_{k-1}^0(t,x,y,s)\,\mathrm{d}s = k_{k-1}^0(t,x,y,Y), \quad \text{on } Y = y = 0. \tag{21b}$$

5.3. Determination of u_l^0 and v_l^0

The Fourier coefficients u_l^0 and v_l^0 are determined in 3 steps. First, from Eqs. (15) with $k = l$, satisfied by u_l^0 and v_l^0, we obtain expressions of u_l^0 and v_l^0, where $\underline{u}_l^0 \in \underline{S}$ and $\underline{v}_l^0 \in \underline{S}$ have to be determined. Afterward, the boundary conditions (21) with $k = l+1$ satisfied by u_{l+1}^0 and v_{l+1}^0 yield boundary conditions satisfied by $\underline{u}_l^0$ and $\underline{v}_l^0$ on $y = 0$. Finally, it follows from Eqs. (15) with $k = l+2$, satisfied by u_{l+2}^0 and v_{l+2}^0, the resolubility conditions (16), which lead to equations satisfied by $\underline{u}_l^0$ and $\underline{v}_l^0$ on $y > 0$.

6. Determination of U_2^0

The equations satisfied by $u_2^0 \in S$, $v_2^0 \in S$, yield $u_2^0 = \underline{u}_2 \in \underline{S}$, $v_2^0 = \underline{v}_2 \in \underline{S}$.
The boundary conditions (20) for $k = 3$ read:

$$\partial_y \underline{u}_2 + \partial_x \underline{v}_2 = 0, \qquad (r-2)\partial_x \underline{u}_2 + r\partial_y \underline{v}_2 = 0, \quad \text{on } y = 0. \tag{22}$$

The resolubility conditions (16) $\underline{H}_3^0 = \underline{K}_3^0 = 0$ yield

$$\partial_{tt}\underline{u}_2 - r\partial_{xx}\underline{u}_2 - (r-1)\partial_{xy}\underline{v}_2 - \partial_{yy}\underline{u}_2 = 0, \tag{23a}$$

$$\partial_{tt}\underline{v}_2 - \partial_{xx}\underline{v}_2 - (r-1)\partial_{xy}\underline{u}_2 - r\partial_{yy}\underline{v}_2 = 0, \quad \text{in } y > 0. \tag{23b}$$

From (22) and (23), we then obtain $\underline{u}_2 = \underline{v}_2 = 0$ and thus $u_2^0 = v_2^0 = 0$.

7. Determination of U_3^0

From $U_2^0 = 0$ and $U_2 \in S^*$ (thus U_2 independent of y) and (9), we get:

$$H_3^0 = \partial_Y\big[(r-1)\partial_x v_3^0 + 2\partial_y u_3^0\big] + \partial_x[\partial_Y u_2 \partial_\theta v_2]^0 + \partial_Y\big[\partial_\theta v_2(\partial_\theta u_3 + \partial_Y v_3 + \partial_x u_2 + \partial_y v_2) + (\partial_\theta v_3 + \partial_x v_2)(\partial_\theta u_2 + \partial_Y v_2)\big]^0.$$

Remark 7.1. For $f = \underline{f} + f^* \in S = \underline{S} + S^*$

$$\int_Y^\infty \partial_Y f(t,x,y,\theta,s)\,\mathrm{d}s = \lim_{Y\to+\infty} f(t,x,y,\theta,Y) - f(t,x,y,\theta,Y) = \underline{f}(t,x,y) - f(t,x,y,\theta,Y) = -f^*(t,x,\theta,Y).$$

Thus

$$\int_Y^\infty H_3^0(t,x,y,s)\,\mathrm{d}s = \int_Y^\infty \partial_x[\partial_Y u_2 \partial_\theta v_2]^0\,\mathrm{d}s - (r-1)\partial_x v_3^{0,*} - 2\partial_y u_3^{0,*} - \big[\partial_\theta v_2(\partial_\theta u_3 + \partial_Y v_3 + \partial_x u_2 + \partial_y v_2) + (\partial_\theta v_3 + \partial_x v_2)(\partial_\theta u_2 + \partial_Y v_2)\big]^0.$$

From the expression of h_3 (10), we then obtain that (21a) for $k=4$ reads

$$\partial_y \underline{u}_3 + \partial_x \underline{v}_3 = (r-2)\partial_x v_3^{0,*} - \int_0^\infty \partial_x[\partial_Y u_2 \partial_\theta v_2]^0 + \partial_y\big[\partial_\theta v_2(\partial_\theta u_2 + \partial_Y v_2)\big]^0\,\mathrm{d}s, \quad \text{on } Y = y = 0. \tag{24}$$

From the expression of $v_3^{0,*}$ (19), Eq. (24) yields:

$$\partial_y \underline{u}_3 + \partial_x \underline{v}_3 = -\frac{2}{r}\int_0^\infty \partial_x[\partial_Y u_2 \partial_\theta v_2]^0\,\mathrm{d}s \quad \text{on } y = 0. \tag{25}$$

The right-hand side term yields a quadratic interaction.

Plugging the expressions of u_2^n and v_2^n (13) and (14),

$$[\partial_Y u_2 \partial_\theta v_2]^0 = \sum_{n\in\mathbb{Z}^*} -|n|^2 q p_2 |\alpha_2(n,t,x)|^2\big[-qp_1 e^{2|n|p_1 Y} - qp_2 e^{2|n|p_2 Y} + \big(q^2 p_1 + p_2\big)e^{|n|(p_1+p_2)Y}\big].$$

Thus with $C_r = -q(p_1p_2 + p_2^2) + q^2 p_1 p_2 + p_2^2$

$$\int_0^\infty [\partial_Y u_2 \partial_\theta v_2]^0\,\mathrm{d}s = \sum_{n\in\mathbb{Z}^*} |n|\frac{q}{p_1+p_2} C_r\big|\alpha_2(n,t,x)\big|^2.$$

From (11) and (12), we get $C_r = -q^3 - qp_2^2 + q^4 + p_2^2 = (1-q)(p_2^2 - q^3) = \frac{c^2}{2}[1 - \frac{c^2}{r} - (1-\frac{c^2}{2})^3]$. c satisfies Eq. (12), therefore $(1-\frac{c^2}{2})^4 = q^4 = p_1^2 p_2^2 = (1-c^2)(1-\frac{c^2}{r})$. Since $1 - c^2 = p_1^2 > 0$, we have $1 - \frac{c^2}{2} > 0$, thus

$$C_r = \frac{c^2}{2}\left[\frac{(1-\frac{c^2}{2})^4}{1-c^2} - \left(1 - \frac{c^2}{2}\right)^3\right] = \frac{c^4}{4}\frac{(1-\frac{c^2}{2})^3}{1-c^2}, \quad C_r > 0.$$

The boundary condition (25) is thus equivalent to Eq. (6a). Similarly, (21b) yields Eq. (6b) (in this case, we sum over $\mathbb{N}$ odd terms with respect to n). The resolubility conditions (16) for $k = 5$, $\underline{H}_4^0 = \underline{K}_4^0 = 0$, yield (5).

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