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Positive gravitational energy in arbitrary dimensions

Énergie gravitationnelle positive en dimension quelconque

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ABSTRACT

We present a streamlined, complete proof, valid in arbitrary space dimension n, and using only spinors on the oriented Riemannian space $(M^n; g)$, of the positive energy theorem in General Relativity.

RÉSUMÉ

On démontre un théorème d'énergie gravitationnelle positive en dimension quelconque utilisant seulement des spineurs liés au groupe *Spin*(n) sur une section d'espace Riemannienne (M^n, g).

Version française abrégée

Un espace-temps Einsteinien est une variété Lorentzienne $(\mathbf{M}^{n+1}, \mathbf{g})$ qui satisfait les équations d'Einstein

$$\mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\mathbf{R} = T_{\alpha\beta};$$

on supposera qu'il satisfait la condition d'énergie dominante, $u_\alpha T^{\alpha\beta}$ temporel pour tout vecteur temporel u. Sur chaque section spatiale M^n la métrique induite g et la courbure extrinsèque K satisfont les contraintes qui s'écrivent dans un repère orthonormé d'axes e_i tangents et e_0 orthogonal à M^n

$$\mathbf{R}_{0j} \equiv \partial_j K_h^h - D_h K^h{}_j = T_{0j}, \tag{1}$$

$$\mathbf{S}_{00} \equiv \frac{1}{2}\left\{R - |K|^2 + (\operatorname{tr} K)^2\right\} = T_{00}. \tag{2}$$

On suppose que M^n est l'union d'un compact W et d'un nombre fini N d'ensembles Ω_I, appelés bouts (ends), diffeomorphes au complément d'une boule de R^n. On utilise une partition lisse f_I, f_K de l'unité sur M^n de supports contenus dans un Ω_I ou un ouvert W_K difféomorphe à une boule de R^n, l'union (finie) des W_K recouvrant W. Tous ces ouverts sont munis de coordonnées locales x^i et de la métrique euclidienne $e \equiv \eta_{ij}\,dx^i\,dx^j$. Un tenseur u sur M^n est une somme de tenseurs $u_I = f_I u$, $u_K = f_K u$. On utilise la métrique euclidienne pour définir les normes de ces tenseurs. Un espace de Banach C^k_β ou de Hilbert $H_{s,\delta}$ d'un tenseur u sur M^n est défini à l'aide du *sup* ou de la somme des normes des tenseurs

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u_I et u_K, des choix différents de partition de l'unité donnent des normes équivalentes. La variété Riemannienne (M^n, g) est dite asymptotiquement euclidienne (A.E.) si

$$h := g - \underline{g} \in H_{s,\delta} \cap C^1_{n-2}, \quad s > \frac{n}{2} + 1,\ \frac{n}{2} - 2 > \delta > -\frac{n}{2}, \tag{3}$$

où $\underline{g}$ est une métrique lisse identique dans chaque Ω_I à la métrique euclidienne e.

La définition des masses gravitationelles m_I et des moments p_I, dits ADM, d'un espace-temps muni d'une section A.E. (M^n, g) et de $K \in H_{s-1,\delta+1}$ provient de la formulation Hamiltonienne des équations d'Einstein, on a dans Ω_I

$$m_I := \lim_{r\to\infty} \frac{1}{2} \int_{S_r^{n-1}} \left(\frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i}\right) n_i \mu_e, \quad r \equiv \left\{\sum_i (x^i)^2\right\}^{\frac{1}{2}}, \tag{4}$$

$$p_I^h := \lim_{r\to\infty} \int_{S_r^{n-1}} P^{ih} n_i \mu_e, \quad P^{ih} := K^{ih} - g^{ih}\operatorname{tr}K. \tag{5}$$

On reprend l'idée spinorielle de Witten pour démontrer, mais en utilisant seulement un spineur sur M^n lié à l'algèbre de Clifford $\mathcal{C}l(n)$, la positivité de la masse d'un espace-temps Einsteinien quand $R \geqslant 0$, donc sous la condition d'énergie dominante, quand M^n est une hypersurface maximale. Une formulation simple liée au moment P, qui ne fait intervenir que la même sorte de spineurs sur M^n, permet de montrer que $m_I \geqslant |p_I|$ donc $m \geqslant |p|$ sans condition sur R ni autre hypothèse sur les sources. Les démonstrations reposent sur un théorème d'existence pour la solution d'une équation de Dirac complétée, elliptique, sur une variété asymptotiquement euclidienne.

1. Introduction

The most elegant and convincing proof of the positive energy theorem is by using spinors, as did Witten[1] in the case $n = 3$ inspired by heuristic works of Deser and Grisaru originating from supergravity. The aim of this Note is to present a streamlined, complete proof, valid in arbitrary space dimension n, and using only spinors on the oriented Riemannian space $(M^n; g)$, without invoking spacetime spinors.

We first give the notations and the definitions we use.

2. Definitions

2.1. Asymptotically Euclidean space

M^n is a smooth manifold union of a compact set W and a finite number of sets Ω_I, diffeomorphic to the complement of a ball in R^n. One covers W by a finite number of open sets W_K each diffeomorphic to a ball in R^n. We denote by x^i local coordinates for a domain Ω_I or W_K. We set $r := \{\sum (x^i)^2\}^{\frac{1}{2}}$ and take $r_0 > 0$ such that $\Omega_I := \{r > r_0\}$, $\Omega_I \cap W_K = \emptyset$ if $r < 2r_0$. We consider a preparation of M^n, i.e. a smooth partition of unity, f_I, f_K, f_K with support in W_k, f_I support in Ω_I and $f_I = 1$ for $r > 2r_0$. The Riemannian metric g is continuous and uniformly bounded above and below in each Ω_I, W_K by constant positive definite quadratic forms. A tensor field u on M^n is written as $u \equiv \sum_I u_I + \sum_K u_K$ with $u_I := f_I u$, $u_K := f_K u$. Norms on spaces of tensor fields are defined through their components in the Ω_I, W_K, each endowed with the Euclidean metric $e := \eta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \equiv \sum (\mathrm{d}x^i)^2$, with pointwise norm $|.|$ and volume element μ_e. We use the Banach and Hilbert spaces C^k_β and $H_{s,\delta}$ with norms

$$\|u\|_{C^k_\beta} \equiv \sup_{I,K}\left\{\sup_{\Omega_I}\left(r^{\beta+k}\left|\underline{D}^k u_I\right|\right), \sup_{W_K}\left|\underline{D}^k u_K\right|\right\}, \quad \underline{D}^k := \frac{\partial^k}{\partial x^{i_1}\cdots\partial x^{i_k}}, \tag{6}$$

$$\|u\|^2_{H_{s,\delta}} := \sum_{I=1,\dots,N} \int_{\Omega_I} \sum_{0\leqslant k\leqslant s} r^{2(k+\delta)}\left|\underline{D}^k u_I\right|^2 \mu_e + \sum_{K=1,\dots,N'} \int_{W_K} \sum_{0\leqslant k\leqslant s} \left|\underline{D}^k u_K\right|^2 \mu_e. \tag{7}$$

Different preparations of M^n give equivalent norms. A Riemannian manifold (M^n, g) is called asymptotically Euclidean (A.E.) if

$$h_I := f_I(g - e) \in H_{s,\delta} \cap C^1_{n-2}, \qquad f_K g \in H_s, \quad s > \frac{n}{2} + 1,\ \frac{n}{2} - 2 > \delta > -\frac{n}{2}. \tag{8}$$

It can be proved (using the fact that $H_{s,\delta}$ is an algebra if $s > \frac{n}{2}, \delta > -\frac{n}{2}$) that an A.E. (M^n, g) admits in each end Ω_I an orthonormal coframe

[1] For references prior to 1983 one can consult my survey on positive energy theorems for les Houches 1983 school reproduced in Y. Choquet-Bruhat [1].

$$\theta^j := a^j_i\, dx^i, \quad a^j_i = \delta^j_i + \frac{1}{2}\lambda^j_i,\ \lambda^j_i \in H_{s,\delta} \cap C^1_{n-2}. \tag{9}$$

In the following, components in the coordinates x^i are underlined. In Ω_I it holds that

$$\underline{g_{ij}} \equiv \sum_h a^h_i a^h_j \equiv \eta_{ij} + \underline{h_{ij}}, \quad \eta_{ij} := \delta^j_i,$$
$$\underline{h_{ij}} \equiv \frac{1}{2}\big(\lambda^j_i + \lambda^i_j\big) + \frac{1}{4}\sum_h \lambda^h_j \lambda^h_i, \quad \lambda^h_j \lambda^h_i \in H_{s,2\delta+\frac{n}{2}} \cap C^1_{2n-4}. \tag{10}$$

The rotation coefficients c^h_{ij} of the coframe θ^h are, with (b^j_i) the matrix inverse of (a^i_j) and ∂_i the Pfaff derivative with respect to θ^i, $d\theta^h \equiv \frac{1}{2}c^h_{ij}\theta^i \wedge \theta^j$,

$$c^h_{ij} \equiv b^k_j \partial_i \lambda^h_k - b^k_i \partial_j \lambda^h_i \equiv \frac{1}{2}\big(\partial_i \lambda^h_j - \partial_j \lambda^h_i\big) + \chi^h_{ij}, \quad \chi^h_{ij} \in H_{s-1,2\delta+1+\frac{n}{2}}. \tag{11}$$

We choose the coframe such that

$$\partial_i\big(\lambda^h_j - \lambda^j_h\big) \in H_{s-1,2\delta+1+\frac{n}{2}},$$

the components $\omega_{i,jh} \equiv \frac{1}{2}(-c^i_{jh} + c^h_{ij} - c^j_{ih})$ of the Riemannian connection ω in the coframe θ^i are then computed to be

$$\omega_{i,hj} \equiv \frac{1}{2}\underline{\partial}_h \underline{h_{ij}} - \frac{1}{2}\underline{\partial}_j \underline{h_{ih}} + \zeta_{i,hj}, \quad \zeta_{i,hj} \in H_{s-1,2\delta+1+\frac{n}{2}}. \tag{12}$$

2.2. Global mass m and linear momentum p

We say that (M^n, g, K) is A.E. if (M^n, g) is A.E. and $K \in H_{s-1,\delta+1} \cap C^0_{n-1}$. The mass m and linear momentum p associated to an end Ω define a spacetime vector $\mathbf{E}$ with components

$$E^0 := m := \lim_{r\to\infty} \frac{1}{2} \int_{S^{n-1}_r} (\underline{\partial}_j \underline{h_{ij}} - \underline{\partial}_i \underline{h_{jj}}) n_i \mu_{\bar{g}}, \qquad E^h := p^h := \lim_{r\to\infty} \int_{S^{n-1}_r} P^{ih} n_i \mu_{\bar{g}}. \tag{13}$$

The uniform bound of n_i and the equivalence of $\mu_{\bar{g}}$ with $r^{n-1}\mu_{S^{n-1}_1}$ show that the limits exist.

We always assume that the constraints (1) and (2) are satisfied on M^n and that T obeys the dominant energy condition.

3. Spinor fields and Dirac operator

The gamma matrices associated with an orthonormal coframe θ^i of g at $x \in M^n$ are linear endomorphisms of a complex vector space S of dimension $p := 2^{[n/2]}$ which satisfy the identities

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{ij} I_p, \quad i, j = 1, \ldots, n,\ I_p \text{ identity matrix.} \tag{14}$$

The γ_i are chosen hermitian, i.e. $\gamma_i = \tilde{\gamma}_i$, as is possible for an $O(n)$ group.

The spinor group $Spin(n)$, double covering of $SO(n)$, can be realized by the group of invertible linear maps Λ of S which satisfy, with $O := (O^j_i)$ an $n \times n$ orthogonal matrix

$$\Lambda\gamma^i\Lambda^{-1} = O^i_j \gamma^j \quad \text{and} \quad \det\Lambda = 1. \tag{15}$$

In a subset Ω_I or W_K with a given field ρ_0 of orthonormal frames a spinor field ψ is represented by a mapping $(x^i) \mapsto \underline{\psi}(x^i) \in S$. Under an $O \in SO(n)$ change of frame, $\rho = O\rho_0$ the spinor ψ becomes represented by $\underline{\psi}' = \Lambda\underline{\psi}$ where some choice has been made for the correspondence between Λ and O. This can be made consistently on M^n if it admits a spin structure; that is, a homomorphism of a $Spin(n)$ principal bundle P_{Spin_n} onto the principal bundle of oriented orthonormal frames. It is a topological property of M^n, the vanishing of its second Stiefel–Whitney class, always true for an orientable M^3. A spinor field on M^n is then a section of a vector bundle $\Psi_{Spin(n)}$ associated with $P_{Spin(n)}$, with base M^n and typical fiber S. To a space of spinors corresponds a space of cospinors, replacing S by the adjoint (complex dual) vector space $\tilde{S}$ and the change of representation by $\underline{\phi'} = \underline{\phi}\Lambda^{-1}$. Using dual frames e_A of S and θ^A of $\tilde{S}$ we have $\psi \equiv \psi^A e_A$, $\phi = \theta^A \phi_A$, $A = 1, \ldots, p$, we denote the duality relation by

$$\phi\psi \equiv (\phi, \psi) := \phi_A \psi^A, \quad \text{a frame independent scalar.}$$

By (15) $\tilde{\psi}$ represented by $\tilde{\psi}_A := (\psi^A)^*$ is a cospinor if ψ is a spinor, $|\psi|^2 \equiv \tilde{\psi}\psi$ is positive definite.

A spin connection σ on (M^n, g) is deduced from an $O(n)$ connection ω by the isomorphism between the Lie algebras of $O(n)$ and $Spin(n)$ obtained by differentiation of (15), it is represented in each domain of the preparation by

$$\sigma_i \equiv \frac{1}{4}\gamma^h\gamma^k\omega_{i,hk}, \quad i = 1, \ldots, n. \tag{16}$$

The covariant derivative of a spinor ψ, resp. cospinor ϕ, is a covariant vector spinor, resp. cospinor, with components in the frames $\theta^i \otimes e_A$, resp. $\theta^i \otimes \theta^A$,

$$(D_i\psi)^A \equiv \partial_i\psi^A + (\sigma_i\psi)^A, \qquad (D_i\phi)_A \equiv \partial_i\phi_A - (\phi\sigma_i)_A.$$

The hermiticity of γ_i and (14) show that $\tilde{\sigma}_i = -\sigma_i$, hence $\widetilde{D_i\psi} \equiv D_i\tilde{\psi}$.

The Riemannian connection together with the spin connection define a first order derivation operator mapping tensor–spinor–cospinor fields into tensor–spinor–cospinor fields with one more covariant index. *The gamma matrices are the components of a vector–spinor–cospinor which has covariant derivative zero.*

The spin curvature ρ is a 2-tensor–spinor–cospinor, image by the mapping of Lie algebras of the curvature tensor of g. The Ricci identity for spinors reads

$$D_iD_j\psi - D_jD_i\psi \equiv \rho_{ij}\psi \quad \text{with } \rho_{ij} := \frac{1}{4}R_{ij,hk}\gamma^h\gamma^k. \tag{17}$$

The Dirac operator on sections of the vector bundle $\Psi(n)$ reads locally

$$\mathcal{D}\psi \equiv \gamma^i D_i\psi \equiv \gamma^i\left(\partial_i\psi + \frac{1}{4}\omega_{i,hk}\gamma^h\gamma^k\psi\right), \quad \text{hence } \widetilde{\mathcal{D}\psi} \equiv D_i\tilde{\psi}\gamma^i. \tag{18}$$

The algebraic Bianchi identity together with (14) and (17) lead to the formula (see, for instance, A. Lichnerowicz [3]):

$$\mathcal{D}^2\psi \equiv \eta^{ij}D_iD_j\psi + \frac{1}{2}\gamma^i\gamma^j\rho_{ij}\psi \equiv \eta^{ij}D_iD_j\psi - \frac{1}{4}R\psi. \tag{19}$$

The Dirac operator is a first order linear operator with principal symbol $(\eta^{ij}\xi_i\xi_j)^{\frac{p}{2}}$, hence elliptic. Weighted Sobolev spaces for spinor fields on a prepared M^n are defined as for tensor fields after setting $\psi = \sum(f_I\psi + f_K\psi)$ and using representations $\underline{\psi}$. A known theorem (Y. Choquet-Bruhat and D. Christodoulou [2]) gives:

Theorem 1. *On an A.E. (M^n, g) the Dirac operator is a Fredholm operator from spinors in $H_{s,\delta}$ to spinors in $H_{s-1,\delta+1}$, it is an isomorphism if injective. The same is true of $\mathcal{D}\psi + f\psi$ if f is a bounded linear map from spinors in $H_{s,\delta}$ to spinors in $H_{s-1,\delta+1}$.*

4. Gravitational mass

We prove for arbitrary $n > 2$ the fundamental fact used by Witten for $n = 3$.

Theorem 2. *Let (M^n, g) be A.E. The mass m of an end Ω_I is equal to*

$$\lim_{r\to\infty}\int_{S_r^{n-1}} \mathcal{U}_0^i n_i\mu_{\bar{g}} = \frac{m}{2}, \quad \mathcal{U}_0^i := \mathcal{R}e\{\tilde{\psi}_0(\eta^{ij} - \gamma^i\gamma^j)\sigma_j\psi_0\}, \tag{20}$$

with S_r^{n-1} the submanifold of the end Ω_I with equation $\{\sum(x^i)^2\}^{\frac{1}{2}} = r$, n_i its unit normal, $\mu_{\bar{g}}$ the volume element induced by g and ψ_0 a spinor constant in Ω_I (i.e. $\frac{\partial\psi_0}{\partial x^i} = 0$) and $|\psi_0| = 1$.

Proof. We first remark that, using $\gamma^i = \tilde{\gamma}^i$ and $\tilde{\sigma}_i = -\sigma_i$, one finds

$$\mathcal{R}e(\tilde{\psi}_0\eta^{ij}\sigma_j\psi_0) \equiv \frac{1}{2}\tilde{\psi}_0\eta^{ij}(\sigma_j + \tilde{\sigma}_j)\psi_0 \equiv 0.$$

The definition (16) of σ_j then implies

$$\mathcal{U}_0^i = \frac{1}{8}\sum_{j,h,k}\tilde{\psi}_0\omega_{j,hk}(\gamma^i\gamma^j\gamma^h\gamma^k - \gamma^h\gamma^k\gamma^j\gamma^i)\psi_0.$$

The property (12) of ω on A.E. (M^n, g) shows that $r^{n-1}(\omega_{j,hk} - \frac{1}{2}\underline{\partial}_h\underline{h_{jk}} + \frac{1}{2}\underline{\partial}_k\underline{h_{jh}})$ tends uniformly to zero as r tends to infinity. The uniform bound of n_i and the equivalence of $\mu_{\bar{g}}$ with $r^{n-1}\mu_{S_1^{n-1}}$ show that the limit in (20) exists. Calculations[2] using (14), (12) and the symmetry of $\underline{\partial}_k\underline{h_{hj}}$ in h and j lead to

[2] Similar to those done by P. Chrusciel in his Krakow lectures on Energy in General Relativity.

$$\lim_{r\to\infty}\int_{S_r^{n-1}} \mathcal{U}_0^i n_i \mu_{\bar g} = \frac{1}{2}\lim_{r\to\infty}\int_{S_r^{n-1}} \omega_{j,ij}|\psi_0|^2 n_i \mu_{\bar g} = \frac{1}{2}m \quad \text{if } |\psi_0|^2 = 1. \qquad \square$$

To study the positivity of the mass one defines a vector $\mathcal{U}^i$ on M^n such that the integrals on S_r^{n-1} of $\mathcal{U}^i$ and $\mathcal{U}_0^i$ have the same limit when $r \to \infty$. The Stokes formula applied to the integral of the of divergence of $\mathcal{U}^i$ will give information on this limit. We set

$$\mathcal{U}^i := \mathcal{R}e\{\tilde\psi(\eta^{ij}D_j\psi - \gamma^i\gamma^j D_j\psi)\}. \tag{21}$$

Lemma 3. *On an A.E. manifold (M^n, g) it holds that*

1. $D_i\mathcal{U}^i \geqslant 0 \quad$ *if $R \geqslant 0$ and $\mathcal{D}\psi = 0$.*
2. *If $\psi = \psi_0 + \psi_1$ with $\underline{\partial_i}\psi_0 = 0$ in Ω_I and $\psi_1 \in H_{s,\delta}$ then in Ω_I*

$$\lim_{r\to\infty}\int_{S_r^{n-1}} \mathcal{U}_0^i n_i \mu_{\bar g} = \lim_{r\to\infty}\int_{S_r^n} \mathcal{U}^i n_i \mu_{\bar g}. \tag{22}$$

Proof. 1. By elementary computation, using $D_i\tilde\psi = \widetilde{D_i\psi}$ and the identity (19) one finds

$$D_i\mathcal{U}^i \equiv |D\psi|^2 - |\mathcal{D}\psi|^2 + \frac{1}{4}R|\psi|^2. \tag{23}$$

Therefore $D_i\mathcal{U}^i \geqslant 0$ if $R \geqslant 0$ and ψ satisfies the equation $\mathcal{D}\psi = 0$.

2. To study the limit of the integral on S_r^{n-1} of $\mathcal{U}^i$ when $\psi = \psi_0 + \psi_1$ we write

$$\mathcal{U}^i = \mathcal{U}_0^i + \frac{1}{2}\mathcal{R}e\{\tilde\psi_0[\gamma^j, \gamma^i]D_j\psi_1 + \tilde\psi_1[\gamma^j, \gamma^i]D_j\psi\}. \tag{24}$$

Hence $D_i\mathcal{U}^i = D_i\mathcal{U}_0^i + D_i\mathcal{V}^i$,

$$\mathcal{V}^i \equiv \frac{1}{2}\mathcal{R}_e\{\tilde\psi_0[\gamma^j, \gamma^i]D_j\psi_1 + \tilde\psi_1[\gamma^j, \gamma^i]D_j\psi\}. \tag{25}$$

Embedding and multiplication properties of Sobolev spaces give

$$\tilde\psi_1[\gamma^j, \gamma^i]D_j\psi \in H_{s,\delta} \times \{C_{n-1}^1 \cap H_{s-1,\delta+1}\} \subset H_{s-1,2\delta+1+\frac{n}{2}} \subset C_\beta^0,$$
$$\beta < 2\delta + 1 + \frac{n}{2} + \frac{n}{2} < 2n - 3.$$

To estimate the other term one remarks that

$$D_i\{\tilde\psi_0[\gamma^j, \gamma^i]D_j\psi_1\} \equiv D_iD_j\{\tilde\psi_0[\gamma^j, \gamma^i]\psi_1\} - D_i\{D_j\tilde\psi_0[\gamma^j, \gamma^i]\psi_1\}, \tag{26}$$

the first parenthesis is an antisymmetric 2-tensor hence its double divergence D_iD_j is identically zero. The second parenthesis is

$$D_j\tilde\psi_0[\gamma^j, \gamma^i]\psi_1 \in C_{n-1}^0 \times H_{s,\delta} \subset C_\beta^0.$$

The Stokes formula implies, with $M_r^n := M^n - \{\Omega_I \cap \sum(x^i)^2 \geqslant r^2\}$

$$\int_{M_r^n} D_i\mathcal{U}^i\mu_g = \int_{S_r^{n-1}} \mathcal{U}^i n_i \mu_{\bar g} = \int_{S_r^{n-1}} (\mathcal{U}_0^i + \mathcal{V}^i) n_i \mu_{\bar g}. \tag{27}$$

The fall off properties found for $\mathcal{V}^i$ complete the proof. $\square$

Lemma 4. *If (M^n, g) is A.E. $R \geqslant 0$ and ψ_0 is a smooth spinor constant in Ω_I and zero in the other ends there exists on M^n a spinor $\psi \equiv \psi_0 + \psi_1$, such that $\mathcal{D}\psi = 0$, $\psi_1 \in H_{s,\delta}$.*

Proof. The hypotheses made on ψ_0 show that $\mathcal{D}\psi_0 \in H_{s-1,\delta+1}$. Theorem 1 implies the existence of ψ_1. $\square$

The lemmas imply that $m \geqslant 0$ if $R \geqslant 0$, that is if (M^n, g) is a maximal submanifold of $(\mathbf{M}^{n+1}, \mathbf{g})$; equivalently, if the pointwise gravitational momentum P on M^n has a vanishing trace. We will now lift this restriction, proving moreover that $m \geqslant |p|$.

5. Positive energy

We define a real vector $\mathcal{P}$ on an A.E. (M^n, g, K) by[3]

$$\mathcal{P}^i := \frac{1}{2}\tilde{\psi}\gamma_h P^{ih}\psi \equiv \frac{1}{2}\tilde{\psi}\big(\gamma_h K^{ih} - \gamma^i\gamma^j\gamma^h K_{jh}\big)\psi, \quad P^{ih} = K^{ih} - \delta^{ih}\operatorname{tr} K.$$

If ψ_0 is as before a smooth spinor constant in one end of M^n and zero in the other ends and ψ is a spinor on M^n such that $\psi - \psi_0 \in C^0_\beta$, $\beta > 0$, then

$$\lim_{r\to\infty}\int_{S_r^{n-1}} \mathcal{P}^i n_i \mu_{\bar{g}} = \frac{1}{2}\tilde{\psi}_0\gamma_h p^h\psi_0, \quad p^h := \lim_{r\to\infty}\int_{S_r^{n-1}} P^{ih} n_i \mu_{\bar{g}}. \tag{28}$$

It is elementary to check using the properties of the γ's that $\gamma_h p^h$ is a hermitian matrix with eigenvalues $\pm|p|$. If we choose for ψ_0 an eigenvector of the eigenvalue $-|p|$ we then have

$$\tilde{\psi}_0\gamma_h p^h\psi_0 = -|\psi_0|^2|p|. \tag{29}$$

To estimate the limit (28), we use again the Stokes formula, with

$$D_i\mathcal{P}^i \equiv \frac{1}{2}D_i\big(\tilde{\psi}\gamma_h P^{ih}\psi\big) \equiv \frac{1}{2}D_i(\tilde{\psi}\gamma_h\psi)P^{ih} + \frac{1}{2}\tilde{\psi}\gamma_h\psi D_i P^{ih}. \tag{30}$$

The momentum constraint (2) gives

$$D_i P^{ih} = -T^h_0.$$

On the other hand, the identity (23) together with the Hamiltonian constraint implies that

$$D_i\mathcal{U}^i \equiv |D\psi|^2 - |\mathcal{D}\psi|^2 + \left(\frac{1}{2}T_{00} + \frac{1}{4}|K|^2 - \frac{1}{4}|\operatorname{tr} K|^2\right)|\psi|^2. \tag{31}$$

We introduce the notations[4]

$$\nabla_i\psi := D_i\psi + \frac{1}{2}\gamma^h K_{ih}\psi, \quad \nabla\!\!\!/\,\psi := \gamma^i\nabla_i \equiv \left(\mathcal{D} + \frac{1}{2}\operatorname{tr} K\right)\psi. \tag{32}$$

Elementary computation using $D_i\eta^{hj} \equiv 0$, $D_i\gamma^h \equiv 0$ gives

$$|\nabla\psi|^2 := \eta^{ij}\widetilde{\nabla_i\psi}\nabla_j\psi \equiv |D\psi|^2 + \frac{1}{2}D_i(\tilde{\psi}\gamma_h\psi)K^{ih} + \frac{1}{4}|K|^2|\psi|^2. \tag{33}$$

The identity (31) can therefore be written after simplification

$$D_i\mathcal{U}^i \equiv |\nabla\psi|^2 - |\mathcal{D}\psi|^2 + \left(\frac{1}{2}T_{00} - \frac{1}{4}|\operatorname{tr} K|^2\right)|\psi|^2 - \frac{1}{2}D_i(\tilde{\psi}\gamma_h\psi)K^{ih}. \tag{34}$$

We deduce from the definition

$$|\nabla\!\!\!/\,\psi|^2 \equiv |\mathcal{D}\psi|^2 + \frac{1}{2}D_i\big(\widetilde{\psi}\gamma^i\psi\big)\operatorname{tr} K + \frac{1}{4}|\operatorname{tr} K|^2|\psi|^2$$

which gives

$$D_i\mathcal{U}^i \equiv |\nabla\psi|^2 - |\nabla\!\!\!/\,\psi|^2 + \frac{1}{2}T_{00}|\psi|^2 - \frac{1}{2}D_i\big(\tilde{\psi}\gamma^h\psi\big)P_{ih}. \tag{35}$$

Lemma 5. *If (M^n, g, K) is A.E. then it holds that*

$$D_i\big(\mathcal{U}^i + \mathcal{P}^i\big) \geqslant 0 \tag{36}$$

if the dominant energy condition holds and ψ satisfies the equation $\nabla\!\!\!/\,\psi = 0$.

[3] Remark that we do not introduce a matrix γ_0.

[4] Note that $\nabla\!\!\!/$ is a linear operator mapping space spinor into space spinors, not the trace on M^n of the covariant derivative of a spacetime spinor.

Proof. The identities (30) and (35) lead to

$$D_i(\mathcal{U}^i + \mathcal{P}^i) \equiv |\nabla\psi|^2 - |\not\nabla\psi|^2 + \mathcal{T}, \quad \mathcal{T} := \frac{1}{2}\left(T_{00}|\psi|^2 - \tilde{\psi}\gamma^h\psi T_{0h}\right), \tag{37}$$

with $\mathcal{T} \geqslant 0$ under the dominant energy condition, because

$$\left|\tilde{\psi}\gamma^h\psi T_{0h}\right| \equiv |\psi|^2\left(\eta^{ih}T_{0i}T_{0h}\right)^{\frac{1}{2}} \leqslant T_{00}|\psi|^2$$

with $\mathcal{T} \geqslant 0$ under the dominant energy condition, because

$$\left|\tilde{\psi}\gamma^h\psi T_{0h}\right| \equiv |\psi|^2\left(\eta^{ih}T_{0i}T_{0h}\right)^{\frac{1}{2}} \leqslant T_{00}|\psi|^2. \qquad \square$$

Lemma 6. *If (M^n, g, K) is A.E., then the equation $\not\nabla\psi = 0$ has a solution $\psi \equiv \psi_0 + \psi_1$, ψ_0 smooth, constant in Ω_I and zero in the other ends, and $\psi_1 \in H_{s,\delta}$.*

Proof. The operator $\not\nabla$ has the same principal part as $\mathcal{D}$, therefore is also elliptic. It maps $H_{s,\delta}$ into $H_{s-1,\delta+1}$. The equation $\not\nabla\psi_1 = -\not\nabla\psi_0 \in H_{s-1,\delta+1}$ has one and only one solution if $\not\nabla$ is injective on $H_{s,\delta}$. To show injectivity[5] we remark that the identity (37) was established without restriction on ψ, starting from the definitions of $\mathcal{U}^i$ and $\mathcal{P}^i$. We make $\psi = \psi_1$ in (37) and integrate it on M^n, the fall off of ψ_1 implies that the divergence gives no contribution, the equation $|\not\nabla\psi_1|^2 = 0$ implies therefore that on M^n, if $\mathcal{T} \geqslant 0$

$$|\nabla\psi_1|^2 = 0, \quad \text{i.e.} \quad D_i\psi_1 + \frac{1}{2}\gamma^h K_{ih}\psi_1 = 0, \quad \text{with } \gamma^h K_{ih} \in H_{s-1,\delta+1}.$$

The Poincaré inequality (see, for instance [1, Appendix 3, Sobolev spaces, p. 541]) in weighted Hilbert spaces leads to $\psi_1 = 0$ if $\psi_1 \in H_{s,\delta}$, $s > \frac{n}{2} + 1$ and $-2 + \frac{n}{2} > \delta > -\frac{n}{2}$. $\square$

The lemmas, after choice of ψ_0 satisfying (29), prove the following theorem:

Theorem 7. *If an Einsteinian spacetime satisfies the dominant energy condition, the energy momentum vector $E^0 = m$, $E^i = p^i$ of each end of an A.E. slice (M^n, g, K) satisfies the inequality*

$$m \geqslant |p|.$$

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References

[1] Y. Choquet-Bruhat, General Relativity and the Einstein Equations, Oxford University Press, 2009.
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[3] A. Lichnerowicz, Champs spinoriels et propagateurs en relativité générale, Bull. Soc. Math. France 92 (1964) 11–100.

[5] See a similar proof in Chrusciel Krakow lecture notes.