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Lie n-racks

n-casiers de Lie

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ABSTRACT

In this Note, we introduce the category of Lie n-racks and generalize several results known on racks. In particular, we show that the tangent space of a Lie n-rack at the neutral element has a Leibniz n-algebra structure.

RÉSUMÉ

Dans cette Note, nous introduisons la catégorie des n-casiers de Lie et nous généralisons plusieurs résultats connus pour les racks. En particulier, nous montrons que l'espace tangent d'un n-casier de Lie en l'élément neutre a une structure de n-algèbre de Leibniz.

1. Introduction and generalities

One of the most important problems in Leibniz algebra theory is the coquecigrue problem (a generalization of Lie's third theorem to Leibniz algebras) which consists of finding a generalization of groups whose tangent algebra structure corresponds to a Leibniz algebra. Loday dubbed these objects "coquecigrues" [7] as no properties were foreseen on them. While attempting to solve this problem, Kinyon [5] showed that the tangent space at the neutral element of a Lie rack has a Leibniz algebra structure.

Meanwhile, Grabowski and Marmo [3] provided in the same order of idea an important connection between Filippov algebras and Nambu–Lie groups. All these ideas suggest a new mathematical structure by extending the binary operation of Lie racks to an n-ary operation. This yields the introduction of Lie n-racks and generalizes Lie racks from the case $n=2$. It turns out that one can extend Kinyon's result to Leibniz n-algebras via Lie n-racks.

Let us recall a few definitions. Given a field $\mathfrak{K}$ of characteristic different to 2, a Leibniz n-algebra [2] is defined as a $\mathfrak{K}$-vector space $\mathfrak{g}$ equipped with an n-linear operation $[-,\dots,-]:\mathfrak{g}^{\otimes n}\to\mathfrak{g}$ satisfying the identity

$$\big[x_1,\dots,x_{n-1},[y_1,y_2,\dots,y_n]\big]=\sum_{i=1}^{n}\big[y_1,\dots,y_{i-1},[x_1,\dots,x_{n-1},y_i],y_{i+1},\dots,y_n\big]. \tag{1}$$

When the *n-ary* operation $[-,\dots,-]$ is antisymmetric in each pair of variables, i.e., $[x_1,x_2,\dots,x,\dots,x,\dots,x_n]=0$ for all $x\in G$, the Leibniz n-algebra becomes a Filippov algebra (more precisely an n-Filippov algebra). Also, a Leibniz 2-algebra is exactly a Leibniz algebra [6, p. 326] and becomes a Lie algebra if the binary operation $[\ ,\ ]$ is skew symmetric. If $\mathfrak{g}$ is a vector

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space endowed with an n-linear operation $\sigma : \mathfrak{g} \times \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathfrak{g}$, then a map $D : \mathfrak{g} \to \mathfrak{g}$ is called a derivation with respect to σ if

$$D\big(\sigma(x_1, \ldots, x_n)\big) = \sum_{i=1}^{n} \sigma\big(x_1, \ldots, D(x_i), \ldots, x_n\big).$$

A lie rack $(R, \circ, 1)$ is a smooth manifold R with a binary operation $\circ$ and a specific element $1 \in R$ such that the following conditions are satisfied:

- $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$;
- for each $x, y \in R$, there exits a unique $a \in R$ such that $x \circ a = y$;
- $1 \circ x = x$ and $x \circ 1 = 1$ for all $x \in R$;
- the operation $\circ : R \times R \to R$ is a smooth mapping.

2. *n*-Racks

Definition 2.1. A left n-rack[1] (right n-racks are defined similarly) $(R, [-, \ldots, -]_R)$ is a set R endowed with an n-ary operation $[-, \ldots, -]_R : R \times R \times \cdots \times R \to R$ such that

(i) $[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_{n-1}]_R]_R = [[x_1, \ldots, x_{n-1}, y_1]_R, \ldots, [x_1, \ldots, x_{n-1}, y_n]_R]_R$. (This is the left distributive property of n-racks);
(ii) For $a_1, \ldots, a_{n-1}, b \in R$, there exists a unique $x \in R$ such that $[a_1, \ldots, a_{n-1}, x]_R = b$.

If in addition there is a distinguish element $1 \in R$, such that

(iii) $[1, \ldots, 1, y]_R = y$ and $[x_1, \ldots, x_{n-1}, 1]_R = 1$ for all $x_1, \ldots, x_{n-1} \in R$, then $(R, [-, \ldots, -]_R, 1)$ is said to be a pointed n-rack. An n-rack is a weak n-quandle if it further satisfies

$$[x, x, \ldots, x, x]_R = x \quad \text{for all } x \in R.$$

An n-rack is an n-quandle if it further satisfies

$$[x_1, x_2, \ldots, x_{n-1}, y]_R = y \quad \text{if } x_i = y \quad \text{for some } i \in \{1, 2, \ldots, n-1\}.$$

An n-quandle (resp. weak n-quandle) is an n-kei (resp. weak n-kei) if it further satisfies

$$\big[x_1, \ldots, x_{n-1}, [x_1, \ldots, x_{n-1}, y]\big] = y \quad \text{for all } x_1, \ldots, x_{n-1}, y \in R.$$

For $n = 2$, one recovers racks, quandles [4] and keis [8]. Note also that n-quandles are also weak n-quandles, but the converse is not true for $n > 2$; see Example 2.3.

Definition 2.2. Let R, R' be n-racks. A function $\alpha : R \to R'$ is said to be a homomorphism of n-racks if

$$\alpha\big([x_1, \ldots, x_n]_R\big) = \big[\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)\big]_{R'} \quad \text{for all } x_1, x_2, \ldots, x_n \in R.$$

We may thus form the category ${}_n pRACK$ of pointed n-racks and pointed n-rack homomorphisms.

Example 2.3. Let $\Gamma := \mathbf{Z}[t^{\pm 1}, s]/(s^2 + ts - s)$. Any Γ-module M endowed with the operation $[-, \ldots, -]_M$ defined by $[x_1, \ldots, x_n]_M = sx_1 + sx_2 + \cdots + sx_{n-1} + tx_n$ is an n-rack that generalizes the Alexander quandle when $s = 1 - t$. Indeed

$$\begin{aligned}
&\big[[x_1, \ldots, x_{n-1}, y_1]_M, \ldots [x_1 \ldots, x_{n-1}, y_n]_M\big]_M \\
&\quad = \left(\sum_{i=1}^{n-1} s(sx_1 + sx_2 + \cdots + sx_{n-1} + ty_i)\right) + t(sx_1 + sx_2 + \cdots + sx_{n-1} + ty_n) \\
&\quad = \big(s^2 + st\big)\left(\sum_{i=1}^{n-1} x_i\right) + ts\left(\sum_{i=1}^{n-1} y_i\right) + t^2 y_n = \big[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_n]_M\big]_M \quad \text{since } s^2 + st = s.
\end{aligned}$$

Therefore (i) is satisfied. One easily checks the axiom (ii). Note that for $t = 1$ and $s = 2$, we obtain an n-rack that is a weak n-kei if n is odd.

[1] 2-racks coincide with Racks. They were introduced in 1959 by G. Wraith and J. Conway [1].

Example 2.4. A group G endowed with the operation $[-,\dots,-]_G$ defined by

$$[x_1,\dots,x_n]_G = x_1x_2\cdots x_{n-1}x_nx_{n-1}^{-1}x_{n-2}^{-1}\cdots x_1^{-1},$$

is a pointed weak n-quandle (pointed by $1\in G$).

This determines a functor $\mathfrak{F}: GROUP \to {}_n pRACK$ from the category of groups to the category of pointed n-racks. The functor $\mathfrak{F}$ is faithful and has a left adjoint $\mathfrak{F}'$ defined as follows: Given a pointed n-rack R, one constructs a group

$$G_R = \langle R\rangle / I$$

where $\langle R\rangle$ stands for the free group on the elements of R and I is the normal subgroup generated by the set

$$\bigl\{\bigl(x_1^{-1}x_2^{-1}\cdots x_{n-1}^{-1}x_n^{-1}x_{n-1}x_{n-2}\cdots x_1\bigr)\bigl([x_1,\dots,x_n]_R\bigr) \text{ with } x_i\in R,\ i=1,2,\dots,n\bigr\}.$$

That $\mathfrak{F}'$ is left adjoint to $\mathfrak{F}$ is a consequence of the following proposition which extends to n-racks a well-known result on the category of racks:

Proposition 2.5. *Let G be a group and let R be an n-rack. For a morphism of n-racks $\alpha: R\to\mathfrak{F}(G)$, there is a unique morphism of groups $\alpha_*: \mathfrak{F}'(R)\to G$ such that the following diagram commutes*:

$$\begin{tikzcd} \mathfrak{F}'(R) \arrow[r, "\alpha_*"] & G \\ R \arrow[u] \arrow[r, "\alpha"] & \mathfrak{F}(G) \arrow[u, "id"'] \end{tikzcd}$$

Proof. By the universal property of free groups, there is a unique morphism of groups $\beta: \langle R\rangle \to G$ such that $\alpha = \beta|_R$. In particular, for all $x_i\in R, i=1,2\dots,n$,

$$\beta\bigl(\bigl(x_1^{-1}x_2^{-1}\cdots x_{n-1}^{-1}x_n^{-1}x_{n-1}x_{n-2}\cdots x_1\bigr)\bigl([x_1,\dots,x_n]_R\bigr)\bigr)$$
$$=\alpha\bigl(\bigl(x_1^{-1}x_2^{-1}\dots x_{n-1}^{-1}x_n^{-1}x_{n-1}x_{n-2}\cdots x_1\bigr)\bigl([x_1,\dots,x_n]_R\bigr)\bigr)=1.$$

The result follows by the universal property of quotient groups. □

Example 2.6. Any rack $(R,\circ,1)$ is also an n-rack under the n-ary operation defined by $[x_1,x_2,\dots,x_n]_R = x_1\circ(x_2\circ(\cdots(x_{n-1}\circ x_n)\cdots))$. This process determines a functor $\mathfrak{G}: pRACK\to {}_n pRACK$, which has as left adjoint, the functor $\mathfrak{G}': {}_n pRACK\to pRACK$ defined as follows: Given a pointed n-rack $(R,[-\cdots-],1)$, then $R^{\times(n-1)}$ endowed with the binary operation

$$(x_1,x_2,\dots,x_{n-1})\circ(y_1,y_2,\dots y_{n-1}) = \bigl([x_1,\dots,x_{n-1},y_1]_R,\dots,[x_1,\dots,x_{n-1},y_{n-1}]_R\bigr)$$

is a rack pointed at $(1,1,\dots,1)$. Let us observe that if R is an n-quandle, then $R^{\times(n-1)}$ is a quandle.

Definition 2.7. Let R be a pointed n-rack and let $S_R=\{f: R\to R,\ \ f \text{ is a bijection}\}$. Then define $\phi: R\times R\times\cdots\times R\to {}_nAut(R)$ by $\phi(x_1,\dots,x_{n-1})(y) = [x_1,\dots,x_{n-1},y]_R$ for all $y\in R$ where ${}_nAut(R)=\{\xi\in S_R/\xi([x_1,\dots,x_n]_R) = [\xi(x_1),\dots,\xi(x_n)]_R\}$. That ϕ is well-defined is a direct consequence of the axiom (ii) of Definition 2.1.

Proposition 2.8. *Let $(R,[-,\dots,-]_R,1)$ be an n-rack, then for all $x_1,\dots,x_{n-1}\in R$, $\phi(x_1,\dots,x_{n-1})$ operates on R by n-rack automorphism, i.e. $\phi(x_1,\dots,x_{n-1})\in {}_nAut(R)$.*

Proof. A direct consequence of the axiom (i) of Definition 2.1. □

3. From Lie *n*-racks to Leibniz *n*-algebras

In this section we define the notion of Lie n-racks and provide a connection with Leibniz n-algebras. Throughout the section, T_1 denotes the tangent functor.

Definition 3.1. A Lie n-rack $(R,[-,\dots,-]_R,1)$ is a smooth manifold R with the structure of a pointed n-rack such that the n-ary operation $[-,\dots,-]_R: R\times R\times\cdots\times R\to R$ is a smooth mapping. For $n=2$, one recovers Lie racks [5].

Example 3.2. Let H be a Lie group. The operation $[x_1,\dots,x_n]_G = x_1x_2\cdots x_{n-1}x_nx_{n-1}^{-1}x_{n-2}^{-1}\cdots x_1^{-1}$ provides H with a Lie n-rack structure.

Example 3.3. Let $(H, \{-\cdots-\})$ be a group endowed with an antisymmetric n-ary operation, and V an H-module. Define the n-ary operation $[-,\dots,-]_R$ on $R := V \times H$ by

$$\big[(u_1, A_1), (u_2, A_2)\cdots(u_n, A_n)\big]_R := \big(\{A_1, \dots, A_n\}u_n, A_1A_2\cdots A_{n-1}A_nA_{n-1}^{-1}A_{n-2}^{-1}\cdots A_1^{-1}\big).$$

Then $(R, [-\cdots-]_R, (0,1))$ is a Lie n-rack.

Theorem 3.4. *Let R be a Lie n-rack and $g := T_1R$. For all $x_1, x_2, \dots, x_{n-1} \in R$, the tangent mapping $\Phi(x_1, x_2, \dots, x_{n-1}) = T_1(\phi(x_1, x_2, \dots, x_{n-1}))$ is an automorphism of $(\mathfrak{g}, [-,\dots,-]_{\mathfrak{g}})$.*

Proof. Since $\phi(x_1, x_2, \dots, x_{n-1})(1) = [x_1, x_2, \dots, x_{n-1}, 1]_R = 1$, we apply the tangent functor T_1 to $\phi(x_1, x_2, \dots, x_{n-1}) : R \to R$ and obtain $\Phi(x_1, x_2, \dots, x_{n-1}) : T_1R \to T_1R$ which is in $GL(T_1R)$ as $\phi(x_1, x_2, \dots, x_{n-1}) \in {}_nAut(R)$ by Proposition 2.8. Now by the left distributive property of n-racks, we have

$$\begin{aligned}&\phi(x_1, x_2, \dots, x_{n-1})\big(\phi(y_1, y_2, \dots, y_{n-1})(y_n)\big)\\ &\quad= \phi\big(\phi(x_1, \dots, x_{n-1})(y_1), \phi(x_1, \dots, x_{n-1})(y_2), \dots, \phi(x_1, \dots, x_{n-1})(y_{n-1})\big)\big(\phi(x_1, \dots, x_{n-1})(y_n)\big)\end{aligned}$$

which successively differentiated at $1 \in R$ with respect to y_n, then y_{n-1}, until y_1 yields to

$$\begin{aligned}&\Phi(x_1, x_2, \dots, x_{n-1})\big([Y_1, Y_2, \dots, Y_n]_{\mathfrak{g}}\big)\\ &\quad= \big[\Phi(x_1, x_2, \dots, x_{n-1})(Y_1), \Phi(x_1, x_2, \dots, x_{n-1})(Y_2), \dots, \Phi(x_1, x_2, \dots, x_{n-1})(Y_n)\big]_{\mathfrak{g}}\end{aligned} \tag{2}$$

for all $Y_1, Y_2, \dots, Y_n \in \mathfrak{g}$. $\square$

Theorem 3.5. *Let R be a Lie n-rack and let $x_1, \dots, x_{n-1} \in R$ corresponding respectively to $X_1, \dots, X_{n-1} \in \mathfrak{g} := T_1R$. Then, the adjoint derivation $ad\{X_1, \dots, X_{n-1}\} : \mathfrak{g} \to gl(\mathfrak{g})$ defined by*

$$ad\{X_1, X_2, \dots, X_{n-1}\}(Y) = [X_1, X_2, \dots, X_{n-1}, Y]_{\mathfrak{g}}$$

is exactly $T_1(\Phi)$.

Proof. From the proof of Theorem 3.4, $\Phi(x_1, x_2, \dots, x_{n-1}) \in GL(\mathfrak{g})$. Also, the mapping $\Phi : R \times R \times \cdots \times R \to GL(\mathfrak{g})$ satisfies $\Phi(1, 1, \dots, 1) = I$, where $I \in GL(\mathfrak{g})$ is the identity. Differentiating Φ at $(1, 1, \dots, 1)$ yields a mapping $T_1(\Phi) : T_1(R \times R \times \cdots \times R) \to gl(\mathfrak{g})$, where $gl(\mathfrak{g})$ is the Lie algebra associated to the Lie group $GL(\mathfrak{g})$. Also differentiating the identity (2) at $(1, 1, \dots, 1)$ with respect to $(x_1, x_2, \dots, x_{n-1})$ yields

$$\big[X_1, \dots, X_{n-1}, [Y_1, Y_2, \dots, Y_n]_{\mathfrak{g}}\big]_{\mathfrak{g}} = \sum_{i=1}^{n}\big[Y_1, \dots, Y_{i-1}, [X_1, \dots, X_{n-1}, Y_i]_{\mathfrak{g}}, Y_{i+1}, \dots, Y_n\big]_{\mathfrak{g}}. \qquad \square$$

Corollary 3.6. *Let R be a Lie n-rack and $\mathfrak{g} := T_1R$. Then there exists an n-linear mapping $[-,\dots,-]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathfrak{g}$ such that $(\mathfrak{g}, [-,\dots,-]_{\mathfrak{g}})$ is a Leibniz n-algebra.*

Proof. From the proofs of Theorems 3.4 and 3.5, it is clear that the n-ary operation $[-,\dots,-]_{\mathfrak{g}}$ is a derivation for itself. $\square$

References

[1] J. Conway, G. Wraith, Unpublished correspondence, 1958.
[2] J.M. Casas, J.L. Loday, T. Pirashvili, Leibniz n-algebras, Forum Math. 14 (2002) 189–207.
[3] J. Grabowski, G. Marmo, On Filippov algebroids and multiplicative Nambu–Poisson structures, Differential Geom. Appl. 12 (2000) 35–50.
[4] D. Joyce, A classifying invariant of knots: the knot quandle, J. Pure Appl. Algebra 23 (1982) 37–65.
[5] K. Kinyon, Leibniz algebra, Lie racks and digroups, J. Lie Theory 17 (2007) 99–111.
[6] J.-L. Loday, Cyclic Homology, Springer-Verlag, Berlin, Heidelberg, New York, 1992.
[7] J.-L. Loday, Une Version Non Commutative des Algebres de Lie : Les Algebres de Leibniz, Enseign. Math. 39 (1993) 269–293.
[8] M. Takasaki, Abstraction of symmetric transformation, Tohoku Math. J. 49 (1942/1943).