C. R. Acad. Sci. Paris, Ser. I 349 (2011) 845–848

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Complex Analysis/Mathematical Analysis

Effective Cartan–Tanaka connections for $\mathscr{C}^6$-smooth strongly pseudoconvex hypersurfaces $M^3 \subset \mathbb{C}^2$

Connections de Cartan–Tanaka effectives pour les hypersurfaces strictement pseudoconvexes $M^3 \subset \mathbb{C}^2$ de classe $\mathscr{C}^6$

Mansour Aghasi [a], Joël Merker [b], Masoud Sabzevari [a]

[a] *Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, Iran*
[b] *Département de mathématiques d'Orsay, bâtiment 425, faculté des sciences, 91405 Orsay cedex, France*

ARTICLE INFO

Article history:
Received 24 June 2011
Accepted 22 July 2011

Presented by Jean-Pierre Demailly

ABSTRACT

Explicit Cartan–Tanaka curvatures, the vanishing of which characterizes sphericity, are provided in terms of the 6-th order jet of a graphing function for a $\mathscr{C}^6$ strongly pseudoconvex hypersurface $M^3 \subset \mathbb{C}^2$.

RÉSUMÉ

Des courbures de Cartan–Tanaka explicites, dont l'annulation identique caractérise la sphéricité, sont fournies en termes du jet d'ordre 6 d'une fonction graphante pour une hypersurface $M^3 \subset \mathbb{C}^2$ de classe $\mathscr{C}^6$ strictement pseudoconvexe.

1. Cartan connection, curvature function and second cohomology of Lie algebras

Definition 1.1. (See [12,4].) Let G be a real Lie group with a closed subgroup H, and let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding real Lie algebras. A *Cartan geometry of type* (G, H) on a $\mathscr{C}^\infty$ manifold M is a principal H-bundle $\pi : \mathscr{P} \to M$ together with a $\mathfrak{g}$-valued '*Cartan connection*' 1-form $\omega : T\mathscr{P} \to \mathfrak{g}$ satisfying:

(i) $\omega_p : T_p\mathscr{P} \to \mathfrak{g}$ is an isomorphism at every $p \in \mathscr{G}$;
(ii) if $R_h(p) := ph$ is the right translation on $\mathscr{G}$ by an $h \in H$, then $R_h^*\omega = \mathrm{Ad}(h^{-1}) \circ \omega$;
(iii) $\omega(H^\dagger) = \mathsf{h}$ for every $\mathsf{h} \in \mathfrak{h}$, where $H^\dagger|_p := \frac{\mathrm{d}}{\mathrm{d}t}|_0(R_{\exp(t\mathsf{h})}(p))$ is the left-invariant vector field on $\mathscr{G}$ associated to h.

Since the associated curvature 2-form $\Omega(X,Y) := \mathrm{d}\omega(X,Y) + [\omega(X), \omega(Y)]_{\mathfrak{g}}$, with $X, Y \in \Gamma(T\mathscr{P})$, vanishes if either X or Y is vertical [12], Ω is fully represented by the *curvature function* $\kappa \in \mathscr{C}^\infty(\mathscr{P}, \Lambda^2(\mathfrak{g}^*/\mathfrak{h}^*) \otimes \mathfrak{g})$ which sends a point $p \in \mathscr{P}$ to the map $\kappa(p) : (\mathfrak{g}/\mathfrak{h}) \wedge (\mathfrak{g}/\mathfrak{h}) \to \mathfrak{g}$ defined by:

$$(\mathsf{x}' \bmod \mathfrak{h}) \wedge (\mathsf{x}'' \bmod \mathfrak{h}) \longmapsto -\Omega_p\big(\omega_p^{-1}(\mathsf{x}'), \omega_p^{-1}(\mathsf{x}'')\big) = -[\mathsf{x}', \mathsf{x}'']_{\mathfrak{g}} + \omega_p\big([\widehat{X}', \widehat{X}'']\big),$$

E-mail addresses: m.aghasi@cc.iut.ac.ir (M. Aghasi), merker@math.u-psud.fr (J. Merker), sabzevari@math.iut.ac.ir (M. Sabzevari).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.07.020

where $\widehat{X} := \omega^{-1}(\mathsf{x})$ is the *constant field* on $\mathscr{P}$ associated to an $\mathsf{x} \in \mathfrak{g}$. Denote then $r := \dim_{\mathbb{R}} \mathfrak{g}$, $n := \dim_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})$ whence $n - r = \dim_{\mathbb{R}} \mathfrak{h}$ and suppose $r \geqslant 2$, $n \geqslant 1$, $n - r \geqslant 1$ so that $\mathfrak{g}$, $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{h}$ are all nonzero. Picking an adapted basis $(\mathsf{x}_k)_{1\leqslant k\leqslant r}$ with $\mathfrak{g} = \mathrm{Span}_{\mathbb{R}}(\mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{x}_{n+1}, \ldots, \mathsf{x}_r)$ and $\mathfrak{h} = \mathrm{Span}_{\mathbb{R}}(\mathsf{x}_{n+1}, \ldots, \mathsf{x}_r)$, one may expand the curvature function by means of $\mathbb{R}$-valued components:

$$\kappa(p) = \sum_{1\leqslant i_1<i_2\leqslant n} \sum_{k=1}^{r} \kappa^k_{i_1,i_2}(p)\mathsf{x}^*_{i_1} \wedge \mathsf{x}^*_{i_2} \otimes \mathsf{x}_k.$$

Lemma 1.2. *(See [2].) For any field $Y^\dagger = \frac{\mathrm{d}}{\mathrm{d}t}|_0 R_{\exp(t\mathsf{y})}$ on $\mathscr{P}$ associated to an arbitrary $\mathsf{y} \in \mathfrak{h}$, one has:*

$$\big(Y^\dagger\kappa\big)(p)\big(\mathsf{x}', \mathsf{x}''\big) = -\big[\mathsf{y}, \kappa(p)\big(\mathsf{x}', \mathsf{x}''\big)\big]_{\mathfrak{g}} + \kappa(p)\big(\big[\mathsf{y}, \mathsf{x}'\big]_{\mathfrak{g}}, \mathsf{x}''\big) + \kappa(p)\big(\mathsf{x}', \big[\mathsf{y}, \mathsf{x}''\big]_{\mathfrak{g}}\big).$$

For any $k \in \mathbb{N}$, consider k-cochains $\mathscr{C}^k := \Lambda^k(\mathfrak{g}^*/\mathfrak{h}^*) \otimes \mathfrak{g}$ and define the *differential* $\partial^k : \mathscr{C}^k \to \mathscr{C}^{k+1}$ by:

$$\begin{aligned}\big(\partial^k\Phi\big)(\mathsf{z}_0, \mathsf{z}_1, \ldots, \mathsf{z}_k) := &\sum_{i=0}^{k} (-1)^i \big[\mathsf{z}_i, \Phi(\mathsf{z}_0, \ldots, \hat{\mathsf{z}}_i, \ldots, \mathsf{z}_k)\big]_{\mathfrak{g}} \\ &+ \sum_{0\leqslant i<j\leqslant k} (-1)^{i+j} \Phi\big([\mathsf{z}_i, \mathsf{z}_j]_{\mathfrak{g}}, \mathsf{z}_0, \ldots, \hat{\mathsf{z}}_i, \ldots, \hat{\mathsf{z}}_j, \ldots, \mathsf{z}_k\big).\end{aligned}$$

Especially for $k = 2$, the cohomology spaces $\mathscr{H}^k := \ker(\partial^k)/\mathrm{im}(\partial^{k-1})$ encode deformations of Lie algebras and are useful for Cartan connections [13,4,6], cf. [1] for an algorithm using Gröbner bases.

Lemma 1.3. *(Bianchi identity [4,6,2].) For any three $\mathsf{x}', \mathsf{x}'', \mathsf{x}''' \in \mathfrak{g}$, one has at every point $p \in \mathscr{P}$:*

$$0 = \big(\partial^2\kappa\big)(p)\big(\mathsf{x}', \mathsf{x}'', \mathsf{x}'''\big) + \sum_{\mathrm{cycl}} \kappa(p)\big(\kappa(p)\big(\mathsf{x}', \mathsf{x}''\big), \mathsf{x}'''\big) + \sum_{\mathrm{cycl}} \big(\widehat{X}'(\kappa)\big)(p)\big(\mathsf{x}'', \mathsf{x}'''\big).$$

When $\mathfrak{g} = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\nu$ is graded as in Tanaka's theory, with $[\mathfrak{g}_{\lambda_1}, \mathfrak{g}_{\lambda_2}]_{\mathfrak{g}} \subset \mathfrak{g}_{\lambda_1+\lambda_2}$ for any $\lambda_1, \lambda_2 \in \mathbb{Z}$ and when $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\nu$, cochains enjoy a natural grading, the second cohomology is graded too: $\mathscr{H}^2 = \bigoplus_{h\in\mathbb{Z}} \mathscr{H}^2_{[h]}$, and the graded Bianchi identities [6,2]:

$$\partial^2_{[h]}(\kappa_{[h]})\big(\mathsf{x}', \mathsf{x}'', \mathsf{x}'''\big) = -\sum_{\mathrm{cycl}} \sum_{h'=1}^{h-1} \big(\kappa_{[h-h']}\big(\mathrm{proj}_{\mathfrak{g}/\mathfrak{h}}\big(\kappa_{[h']}\big(\mathsf{x}', \mathsf{x}''\big)\big), \mathsf{x}'''\big)\big) - \sum_{\mathrm{cycl}} \big(\widehat{X}'\kappa_{[h+|\mathsf{x}'|]}\big)\big(\mathsf{x}'', \mathsf{x}'''\big)$$

show that the lowest order nonvanishing curvature must be ∂-closed, and more generally, any homogeneous curvature component is determined by the lower components up to a ∂-closed component.

2. Geometry-preserving deformations of the Heisenberg sphere $\mathbb{H}^3 \subset \mathbb{C}^2$

Let now $M^3 \subset \mathbb{C}^2$ be a *local* strongly pseudoconvex $\mathscr{C}^6$-smooth real 3-dimensional hypersurface, represented in coordinates $(z, w) = (x + iy, u + iv)$ as the graph (for background, see [7–11]):

$$v = \varphi(x, y, u) = x^2 + y^2 + \mathrm{O}(3)$$

of a certain real-valued $\mathscr{C}^6$ function φ defined in a neighborhood of the origin in $\mathbb{R}^3$. Its complex tangent bundle $T^cM = \mathrm{Re}\, T^{0,1}M$ is generated by the two vector fields:

$$H_1 := \frac{\partial}{\partial x} + \left(\frac{\varphi_y - \varphi_x\varphi_u}{1 + \varphi_u^2}\right)\frac{\partial}{\partial u} \quad \text{and} \quad H_2 := \frac{\partial}{\partial y} + \left(\frac{-\varphi_x - \varphi_y\varphi_u}{1 + \varphi_u^2}\right)\frac{\partial}{\partial u},$$

which make a frame joint with the third, Levi form-type Lie-bracket:

$$\begin{aligned}T := \frac{1}{4}[H_1, H_2] = \bigg(\frac{1}{4}\frac{1}{(1+\varphi_u^2)^2}\big\{&-\varphi_{xx} - \varphi_{yy} - 2\varphi_y\varphi_{xu} - \varphi_x^2\varphi_{uu} + 2\varphi_x\varphi_{yu} - \varphi_y^2\varphi_{uu} \\ &+ 2\varphi_y\varphi_u\varphi_{yu} + 2\varphi_x\varphi_u\varphi_{xu} - \varphi_u^2\varphi_{xx} - \varphi_u^2\varphi_{yy}\big\}\bigg)\frac{\partial}{\partial u}.\end{aligned}$$

Such M^3's are *geometry-preserving* deformations of the *Heisenberg sphere* $\mathbb{H}^3$: $v = x^2 + y^2$. It is known that the Lie algebra $\mathfrak{hol}(\mathbb{H}^3) := \{\mathsf{X} = Z(z, w)\frac{\partial}{\partial z} + W(z, w)\frac{\partial}{\partial w}:\ \mathsf{X} + \overline{\mathsf{X}} \text{ tangent to } \mathbb{H}^3\}$ of infinitesimal CR automorphisms of the Heisenberg sphere $\mathbb{H}^3$ in $\mathbb{C}^2$ is 8-dimensional and generated by:

$$\mathsf{T} := \partial_w, \qquad \mathsf{H}_1 := \partial_z + 2iz\partial_w, \qquad \mathsf{H}_2 := i\partial_z + 2z\partial_w, \qquad \mathsf{D} := z\partial_z + 2w\partial_w, \qquad \mathsf{R} := iz\partial_z,$$
$$\mathsf{I}_1 := \big(w + 2iz^2\big)\partial_z + 2izw\partial_w, \qquad \mathsf{I}_2 := \big(iw + 2z^2\big)\partial_z + 2zw, \qquad \mathsf{J} := zw\partial_z + w^2\partial_w.$$

Recently, Ezhov, McLaughlin and Schmalz published in the *Notices of the AMS* an expository article [6] in which they reconstruct — within Tanaka's framework and assuming that M is $\mathscr{C}^\omega$ — Cartan's connection [5] valued the eight-dimensional *abstract* real Lie algebra:

$$\mathfrak{g} := \mathbb{R}\mathsf{t} \oplus \mathbb{R}\mathsf{h}_1 \oplus \mathbb{R}\mathsf{h}_2 \oplus \mathbb{R}\mathsf{d} \oplus \mathbb{R}\mathsf{r} \oplus \mathbb{R}\mathsf{i}_1 \oplus \mathbb{R}\mathsf{i}_2 \oplus \mathbb{R}\mathsf{j} \quad (\text{with } \mathfrak{h} := \mathbb{R}\mathsf{d} \oplus \mathbb{R}\mathsf{r} \oplus \mathbb{R}\mathsf{i}_1 \oplus \mathbb{R}\mathsf{i}_2 \oplus \mathbb{R}\mathsf{j})$$

spanned by some eight abstract vectors enjoying the same commutator table as $\mathsf{T}, \dots, \mathsf{J}$ (cf. also [3] for a similar approach). A natural Tanaka grading is: $\mathfrak{g}_{-2} = \mathbb{R}\mathsf{t}$; $\mathfrak{g}_{-1} = \mathbb{R}\mathsf{h}_1 \oplus \mathbb{R}\mathsf{h}_2$; $\mathfrak{g}_0 = \mathbb{R}\mathsf{d} \oplus \mathbb{R}\mathsf{r}$; $\mathfrak{g}_1 = \mathbb{R}\mathsf{i}_1 \oplus \mathbb{R}\mathsf{i}_2$; $\mathfrak{g}_2 = \mathbb{R}\mathsf{j}$. By performing the above choice $\{H_1, H_2, T\}$ of an initial frame for TM which is explicit in terms of the graphing function $\varphi(x, y, u)$, we deviate from the initial normalization made in [6] (with a more geometric-minded approach), since our computational objective is to provide a Cartan–Tanaka connection all elements of which are completely effective in terms of $\varphi(x, y, u)$ — assuming only $\mathscr{C}^6$-smoothness of M.

Call Υ the numerator of $T = \frac{1}{4}[H_1, H_2] = \frac{1}{4}\frac{\Upsilon}{\Delta^2}\frac{\partial}{\partial u}$, allow the two notational coincidences: $x_1 \equiv x$, $x_2 \equiv y$; introduce the two length-three brackets:

$$[H_i, T] = \frac{1}{4}\big[H_i, [H_1, H_2]\big] =: \Phi_i T \quad (i = 1, 2),$$

which are both multiples of T by means of two functions $\Phi_i := \frac{A_i}{\Delta^2 \Upsilon}$; lastly, introduce furthermore the H_k-iterated derivatives of the functions Φ_i up to order 3, where $i, k_1, k_2, k_3 = 1, 2$:

$$H_{k_1}(\Phi_i) = \frac{A_{i,k_1}}{\Delta^4 \Upsilon^2}, \qquad H_{k_2}\big(H_{k_1}(\Phi_i)\big) = \frac{A_{i,k_1,k_2}}{\Delta^6 \Upsilon^3}, \qquad H_{k_3}\big(H_{k_2}\big(H_{k_1}(\Phi_i)\big)\big) = \frac{A_{i,k_1,k_2,k_3}}{\Delta^8 \Upsilon^4}.$$

Proposition 2.1. *(See [2].) All the numerators appearing above are explicitly given by*:

$$\begin{aligned}
A_i &:= \Delta^2 \Upsilon_{x_i} + \Delta(-2\Delta_{x_i}\Upsilon + \Lambda_i \Upsilon_u - \Upsilon \Lambda_{i,u}) - \Lambda_i \Upsilon \Delta_u, \\
A_{i,k_1} &:= \Delta^2(\Upsilon A_{i,x_{k_1}} - \Upsilon_{x_{k_1}} A_i) + \Delta(-2\Delta_{x_{k_1}}\Upsilon A_i + \Upsilon \Lambda_{k_1} A_{i,u} - \Upsilon_u \Lambda_{k_1} A_i) - 2\Delta_u \Upsilon \Lambda_{k_1} A_i, \\
A_{i,k_1,k_2} &:= \Delta^2(\Upsilon A_{i,k_1,x_{k_2}} - 2\Upsilon_{x_{k_2}} A_{i,k_1}) + \Delta(-3\Delta_{x_{k_2}}\Upsilon A_{i,k_1} + \Upsilon \Lambda_{k_2} A_{i,k_1,u} - 2\Upsilon_u \Lambda_{k_2} A_{i,k_1}) - 3\Delta_u \Upsilon \Lambda_{k_2} A_{i,k_1}, \\
A_{i,k_1,k_2,k_3} &= \Delta^2(\Upsilon A_{i,k_1,k_2,x_{k_3}} - \Upsilon_{x_{k_3}} A_{i,k_1,k_2}) + \Delta(-6\Delta_{x_{k_3}}\Upsilon A_{i,k_1,k_2} + \Upsilon \Lambda_{k_3} A_{i,k_1,k_2,u} \\
&\quad - 3\Upsilon_u \Lambda_{k_3} A_{i,k_1,k_2}) - 6\Delta_u \Upsilon \Lambda_{k_3} A_{i,k_1,k_2}.
\end{aligned}$$

Furthermore, these iterated derivatives identically satisfy $H_2(\Phi_1) \equiv H_1(\Phi_2)$ and:

$$\begin{aligned}
0 &\equiv -H_1\big(H_2\big(H_1(\Phi_2)\big)\big) + 2H_2\big(H_1\big(H_1(\Phi_2)\big)\big) - H_2\big(H_2\big(H_1(\Phi_1)\big)\big) - \Phi_2 H_1\big(H_2(\Phi_1)\big) + \Phi_2 H_2\big(H_1(\Phi_1)\big), \\
0 &\equiv -H_2\big(H_1\big(H_1(\Phi_2)\big)\big) + 2H_1\big(H_2\big(H_1(\Phi_2)\big)\big) - H_1\big(H_1\big(H_2(\Phi_2)\big)\big) - \Phi_1 H_2\big(H_1(\Phi_2)\big) + \Phi_1 H_1\big(H_2(\Phi_2)\big), \\
0 &\equiv -H_1\big(H_1\big(H_1(\Phi_2)\big)\big) + 2H_1\big(H_2\big(H_1(\Phi_1)\big)\big) - H_2\big(H_1\big(H_1(\Phi_1)\big)\big) + \Phi_1 H_1\big(H_1(\Phi_2)\big) - \Phi_1 H_2\big(H_1(\Phi_1)\big), \\
0 &\equiv -H_2\big(H_2\big(H_1(\Phi_2)\big)\big) + 2H_2\big(H_1\big(H_2(\Phi_2)\big)\big) - H_1\big(H_2\big(H_2(\Phi_2)\big)\big) + \Phi_2 H_2\big(H_1(\Phi_2)\big) - \Phi_2 H_1\big(H_2(\Phi_2)\big).
\end{aligned}$$

3. Explicit Cartan–Tanaka connection

Theorem 3.1. *(See [2].) Associated to such an $M^3 \subset \mathbb{C}^2$, there is a unique $\mathfrak{g}$-valued Cartan connection which is normal and regular in the sense of Tanaka. Its curvature function reduces to*:

$$\begin{aligned}
\kappa(p) &= \kappa_{i_1}^{h_1 t}(p)\mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 + \kappa_{i_2}^{h_1 t}(p)\mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \kappa_{i_1}^{h_2 t}(p)\mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_1 \\
&\quad + \kappa_{i_2}^{h_2 t}(p)\mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{i}_2 + \kappa_{j}^{h_1 t}(p)\mathsf{h}_1^* \wedge \mathsf{t}^* \otimes \mathsf{j} + \kappa_{j}^{h_2 t}(p)\mathsf{h}_2^* \wedge \mathsf{t}^* \otimes \mathsf{j},
\end{aligned}$$

where the two main curvature coefficients, having homogeneity 4, are of the form:

$$\kappa_{i_1}^{h_1 t}(p) = -\mathbf{\Delta}_1 c^4 - 2\mathbf{\Delta}_4 c^3 d - 2\mathbf{\Delta}_4 c d^3 + \mathbf{\Delta}_1 d^4 \quad \textit{and} \quad \kappa_{i_2}^{h_1 t}(p) = -\mathbf{\Delta}_4 c^4 + 2\mathbf{\Delta}_1 c^3 d + 2\mathbf{\Delta}_1 c d^3 + \mathbf{\Delta}_4 d^4,$$

in which the two functions $\Delta_{\mathbf{1}}$ and $\Delta_{\mathbf{4}}$ of only the three variables (x, y, u) are explicitly *given by*:

$$\begin{aligned}\mathbf{\Delta}_1 = \frac{1}{384}\big[&H_1\big(H_1\big(H_1(\Phi_1)\big)\big) - H_2\big(H_2\big(H_2(\Phi_2)\big)\big) + 11H_1\big(H_2\big(H_1(\Phi_2)\big)\big) - 11H_2\big(H_1\big(H_2(\Phi_1)\big)\big)\\
&+ 6\Phi_2H_2\big(H_1(\Phi_1)\big) - 6\Phi_1H_1\big(H_2(\Phi_2)\big) - 3\Phi_2H_1\big(H_1(\Phi_2)\big) + 3\Phi_1H_2\big(H_2(\Phi_1)\big)\\
&- 3\Phi_1H_1\big(H_1(\Phi_1)\big) + 3\Phi_2H_2\big(H_2(\Phi_2)\big) - 2\Phi_1H_1(\Phi_1) + 2\Phi_2H_2(\Phi_2)\\
&- 2(\Phi_2)^2H_1(\Phi_1) + 2(\Phi_1)^2H_2(\Phi_2) - 2(\Phi_2)^2H_2(\Phi_2) + 2(\Phi_1)^2H_1(\Phi_1)\big],\end{aligned}$$

$$\begin{aligned}\mathbf{\Delta}_4 = \frac{1}{384}\big[&-3H_2\big(H_1\big(H_2(\Phi_2)\big)\big) - 3H_1\big(H_2\big(H_1(\Phi_1)\big)\big) + 5H_1\big(H_2\big(H_2(\Phi_2)\big)\big) + 5H_2\big(H_1\big(H_1(\Phi_1)\big)\big)\\
&+ 4\Phi_1H_1\big(H_1(\Phi_2)\big) + 4\Phi_2H_2\big(H_1(\Phi_2)\big) - 3\Phi_2H_1\big(H_1(\Phi_1)\big) - 3\Phi_1H_2\big(H_2(\Phi_2)\big)\\
&- 7\Phi_2H_1\big(H_2(\Phi_2)\big) - 7\Phi_1H_2\big(H_1(\Phi_1)\big) - 2H_1(\Phi_1)H_1(\Phi_2) - 2H_2(\Phi_2)H_2(\Phi_1)\\
&+ 4\Phi_1\Phi_2H_1(\Phi_1) + 4\Phi_1\Phi_2H_2(\Phi_2)\big],\end{aligned}$$

and where the remaining four secondary curvature coefficients are given by:

$$\kappa_{i_1}^{h_2t} = \kappa_{i_2}^{h_1t}, \qquad \kappa_{i_2}^{h_2t} = -\kappa_{i_1}^{h_1t}, \qquad \kappa_j^{h_1t} = \widehat{H}_1\big(\kappa_{i_2}^{h_2t}\big) - \widehat{H}_2\big(\kappa_{i_2}^{h_1t}\big), \qquad \kappa_j^{h_2t} = -\widehat{H}_1\big(\kappa_{i_1}^{h_2t}\big) + \widehat{H}_2\big(\kappa_{i_1}^{h_1t}\big).$$

Corollary 3.2. *A $\mathscr{C}^\omega$-smooth strongly pseudoconvex local hypersurface $M^3 \subset \mathbb{C}^2$ is biholomorphic to $\mathbb{H}^3$, namely is spherical, if and only if $0 \equiv \mathbf{\Delta}_1 \equiv \mathbf{\Delta}_4$, identically as functions of (x, y, u).*

Acknowledgements

One year ago, Gerd Schmalz kindly provided us with a pdf copy of the accepted version of [6], and this was of great help during the preparation of the presently announced memoir [2].

References

[1] M. Aghasi, B. Alizadeh, J. Merker, M. Sabzevari, A Gröbner-bases algorithm for the computation of the cohomology of Lie (super) algebras, http://arxiv.org/abs/1104.5300, 23 pp.
[2] M. Aghasi, J. Merker, M. Sabzevari, Effective Cartan–Tanaka connections on $\mathscr{C}^6$ strongly pseudoconvex hypersurfaces $M^3 \subset \mathbb{C}^2$, http://ariv.org/abs/1104.1509, 113 pp.
[3] V.K. Beloshapka, V. Ezhov, G. Schmalz, Canonical Cartan connection and holomorphic invariants on Engel CR manifolds, J. Math. Phys. 14 (2007) 121–133 (in Russian).
[4] A. Čap, H. Schichl, Parabolic geometries and canonical Cartan connections, Hokkaido Math. J. 29 (2000) 453–505.
[5] É. Cartan, Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes I, Ann. Math. Pures Appl. 4 (1932) 17–90;
É. Cartan, Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes II, Ann. Sc. Norm. Super. Pisa 2 (1932) 333–354.
[6] V. Ezhov, B. McLaughlin, G. Schmalz, From Cartan to Tanaka: getting real in the complex world, Notices of the AMS 58 (2011) 20–27.
[7] J. Merker, On the partial algebraicity of holomorphic mappings between two real algebraic sets, Bull. Soc. Math. France 129 (2001) 547–591.
[8] J. Merker, On the local geometry of generic submanifolds of $\mathbb{C}^n$ and the analytic reflection principle, J. Math. Sci. (N. Y.) 125 (2005) 751–824.
[9] J. Merker, Lie symmetries and CR geometry, J. Math. Sci. (N. Y.) 154 (2008) 817–922.
[10] J. Merker, Nonrigid spherical real analytic hypersurfaces in $\mathbb{C}^2$, Complex Var. Elliptic Equ. 55 (12) (2010) 1155–1182.
[11] J. Merker, E. Porten, Holomorphic extension of CR functions, envelopes of holomorphy and removable singularities, Int. Math. Res. Surv. (2006), Article ID 28295, 287 pp.
[12] R.W. Sharpe, Differential Geometry, Cartan's Generalization of Klein's Erlangen Program, Springer, Berlin, 1997.
[13] N. Tanaka, On differential systems, graded Lie algebras and pseudo-groups, J. Math. Kyoto Univ. 10 (1970) 1–82.