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On the adjoint representation of $\mathfrak{sl}_n$ and the Fibonacci numbers

Sur la représentation adjointe de $\mathfrak{sl}_n$ et les nombres de Fibonacci

Pamela E. Harris

University of Wisconsin, Milwaukee Department of Mathematical Sciences, P.O. Box 0413, Milwaukee, WI 53201, USA

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ABSTRACT

We decompose the adjoint representation of $\mathfrak{sl}_{r+1} = \mathfrak{sl}_{r+1}(\mathbb{C})$ by a purely combinatorial approach based on the introduction of a certain subset of the Weyl group called the *Weyl alternation set* associated to a pair of dominant integral weights. The cardinality of the Weyl alternation set associated to the highest root and zero weight of $\mathfrak{sl}_{r+1}$ is given by the rth Fibonacci number. We then obtain the exponents of $\mathfrak{sl}_{r+1}$ from this point of view.

RÉSUMÉ

Nous décomposons la représentation adjointe de $\mathfrak{sl}_{r+1} = \mathfrak{sl}_{r+1}(\mathbb{C})$ par une approche purement combinatoire basée sur l'introduction d'un certain sous-ensemble du groupe de Weyl appelé *Weyl alternation set* associé à une paire de poids intégraux dominants. La cardinalité de *Weyl alternation set* associé à la plus haute racine et au poids zéro de $\mathfrak{sl}_{r+1}$ est donnée par le nombre rth de Fibonacci. Nous obtenons alors les exposants de $\mathfrak{sl}_{r+1}$ de ce point de vue.

1. Introduction

Let G be a simple linear algebraic group over $\mathbb{C}$, and T a maximal algebraic torus in G of dimension r. Let B, $T \subseteq B \subseteq G$, be a choice of Borel subgroup. Then let $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{b}$ denote the Lie algebras of G, T, and B respectively. Let Φ be the set of roots corresponding to $(\mathfrak{g}, \mathfrak{h})$, and let $\Phi^+ \subseteq \Phi$ be the choice of positive roots with respect to $\mathfrak{b}$. Let $P(\mathfrak{g})$ be the integral weights with respect to $\mathfrak{h}$, and let $P_+(\mathfrak{g})$ be the dominant integral weights. The theorem of the highest weight asserts that any finite dimensional complex irreducible representation of $\mathfrak{g}$ is equivalent to a highest weight representation with dominant integral highest weight λ, denoted by $L(\lambda)$. For good general references see [2,3].

Let $W = \mathrm{Norm}_G(T)/T$ denote the Weyl group corresponding to G and T, and for any $w \in W$, let $\ell(w)$ denote the length of w. Kostant's partition function is the non-negative integer valued function, $\wp$, defined on $\mathfrak{h}^*$ by $\wp(\xi) =$ number of ways ξ may be written as a non-negative integral sum of positive roots, for $\xi \in \mathfrak{h}^*$.

An area of interest in combinatorial representation theory is finding the multiplicity of a weight μ in $L(\lambda)$. One way to compute this multiplicity, denoted $m(\lambda, \mu)$, is by Kostant's weight multiplicity formula [4]:

$$m(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(w)} \wp\big(\sigma(\lambda + \rho) - (\mu + \rho)\big), \tag{1}$$

E-mail addresses: peharris@uwm.edu, pamela.harris@marquette.edu.

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where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. One complication in using (1) to compute multiplicities is that closed formulas for the value of Kostant's partition function are not known in much generality. A second complication concerns the exponential growth of the Weyl group order as $r \to \infty$. In practice, as noted in [1], most terms in Kostant's weight multiplicity formula are zero and hence do not contribute to the overall multiplicity. With the aim of describing the contributing terms in (1), we give the following definition.

Definition 1.1. For λ, μ dominant integral weights of $\mathfrak{g}$ define the Weyl alternation set to be $\mathcal{A}(\lambda, \mu) = \{\sigma \in W \mid \wp(\sigma(\lambda+\rho) - (\mu+\rho)) > 0\}$.

Thus $\sigma \in \mathcal{A}(\lambda,\mu)$ if and only if $\sigma(\lambda+\rho) - (\mu+\rho)$ is a linear combination with non-negative integral coefficients of positive roots.

The purpose of this short Note is to demonstrate that the sets $\mathcal{A}(\lambda,\mu)$ are combinatorially interesting. To this end, we let $\mathfrak{g} = \mathfrak{sl}_{r+1}$ and prove:

Theorem 1.2. *If $r \geqslant 1$ and $\tilde{\alpha}$ is the highest root of $\mathfrak{sl}_{r+1}$, then $|\mathcal{A}(\tilde{\alpha}, 0)| = F_r$, where F_r denotes the rth Fibonacci number.*

This result gives rise to a (new) combinatorial identity associated to a Cartan subalgebra of $\mathfrak{sl}_{r+1}$, which we present in Section 3. The non-zero weights, μ, of $\mathfrak{sl}_{r+1}$ are considered in Section 4 from the same point of view.

2. The zero weight space

Let $r \geqslant 1$, and $n = r+1$. Let $G = SL_n(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, and[1] $\mathfrak{h} = \{\mathrm{diag}[a_1,\dots,a_n] \mid a_1,\dots,a_n \in \mathbb{C}, \sum_{i=1}^n a_i = 0\}$ be a fixed choice of Cartan subalgebra. Let $\mathfrak{b}$ denote the set of $n \times n$ upper triangular complex matrices with trace zero. For $1 \leqslant i \leqslant n$, define the linear functionals $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ by $\varepsilon_i(H) = a_i$, for any $H = \mathrm{diag}[a_1,\dots,a_n] \in \mathfrak{h}$. The Weyl group, W, is isomorphic to S_n, the symmetric group on n letters, and acts on $\mathfrak{h}^*$ by permutations of $\varepsilon_1,\dots,\varepsilon_n$.

For each $1 \leqslant i \leqslant r$, let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then the set of simple and positive roots corresponding to $(\mathfrak{g},\mathfrak{b})$ are $\Delta = \{\alpha_1,\dots,\alpha_r\}$, and $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}$ respectively. The highest root is $\tilde{\alpha} = \varepsilon_1 - \varepsilon_n = \alpha_1 + \cdots + \alpha_r$.

Theorem 2.1. *Let $\sigma \in S_n$. Then $\sigma \in \mathcal{A}(\tilde{\alpha}, 0)$ if and only if $\sigma(1) = 1$, $\sigma(n) = n$, and $|\sigma(i) - i| \leqslant 1$, for all $1 \leqslant i \leqslant n$.*

The following propositions are used in the proof of Theorem 2.1:

Proposition 2.2. *Let $\sigma \in S_n$. Then σ is a product of commuting neighboring transpositions if and only if $|\sigma(i) - i| \leqslant 1$, for all $1 \leqslant i \leqslant n$.*

Proposition 2.3. *Let $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$, where $i_1,\dots,i_k$ are non-consecutive integers between 2 and $r-1$. Then $\sigma(\tilde{\alpha}+\rho) - \rho = (\varepsilon_1 - \varepsilon_{i_1}) + (\varepsilon_{i_1+1} - \varepsilon_{i_2}) + \cdots + (\varepsilon_{i_k+1} - \varepsilon_n)$ is a combination of positive roots.*

Proof of Theorem 2.1.
($\Rightarrow$) A proof by contradiction shows $\sigma(1) = 1$, and $\sigma(n) = n$, while an induction argument shows $|i - \sigma^{-1}(i)| \leqslant 1$, for all $1 < i < n$. Then Proposition 2.2 implies $\sigma^{-1} = \sigma$, and thus $|i - \sigma(i)| \leqslant 1$, for all $1 < i < n$.
($\Leftarrow$) Proposition 2.3 implies that reciprocally such an element belongs to $\mathcal{A}(\tilde{\alpha}, 0)$. □

Definition 2.4. The Fibonacci numbers are defined in [7] by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$, and $F_1 = F_2 = 1$.

Proof of Theorem 1.2. The theorem follows from Theorem 2.1, and the fact that if $m \geqslant 1$, then $|\{\sigma \in S_m\colon\ |\sigma(i) - i| \leqslant 1$, for all $1 \leqslant i \leqslant m\}| = F_{m+1}$. □

3. A *q*-analog

The q-analog of Kostant's partition function is the polynomial valued function, $\wp_q$, defined on $\mathfrak{h}^*$ by $\wp_q(\xi) = c_0 + c_1 q + \cdots + c_k q^k$, where c_j = number of ways to write ξ as a non-negative integral sum of exactly j positive roots, for $\xi \in \mathfrak{h}^*$. In [6], Lusztig introduced the q-analog of Kostant's weight multiplicity formula:

$$m_q(\lambda,\mu) = \sum_{\sigma\in W} (-1)^{\ell(\sigma)} \wp_q\big(\sigma(\lambda+\rho) - (\mu+\rho)\big).$$

[1] $\mathrm{diag}[a_1,\dots,a_n]$ is the diagonal $n \times n$ matrix whose entries are $a_1,\dots,a_n$.

Given $\mathfrak{g}$, a simple Lie algebra of rank r with highest root $\tilde{\alpha}$, it is known that $m_q(\tilde{\alpha}, 0) = \sum_{i=1}^{r} q^{e_i}$, where $e_1, \ldots, e_r$ are the exponents of $\mathfrak{g}$, [5]. The following results and identities together with the Weyl alternation set $\mathcal{A}(\tilde{\alpha}, 0)$ lead to a combinatorial proof of this result for $\mathfrak{sl}_{r+1}$, whose exponents are $1, 2, \ldots, r$.

Lemma 3.1. *The cardinality of the set* $\{\sigma \in \mathcal{A}(\tilde{\alpha}, 0)\colon \ \ell(\sigma) = k\}$ *is* $\binom{r-1-k}{k}$, *and* $\max\{\ell(\sigma)\colon \ \sigma \in \mathcal{A}(\tilde{\alpha}, 0)\} = \lfloor \frac{r-1}{2} \rfloor$.

Proposition 3.2. *If* $\sigma \in \mathcal{A}(\tilde{\alpha}, 0)$, *then* $\wp_q(\sigma(\tilde{\alpha} + \rho) - \rho) = q^{1+\ell(\sigma)}(1+q)^{r-1-2\ell(\sigma)}$.

Proof. Let $\sigma \in \mathcal{A}(\tilde{\alpha}, 0)$. An induction argument on $\ell(\sigma)$, shows that if $i \geqslant 0$, then, $c_{\ell(\sigma)+1+i}$, the coefficient of $q^{\ell(\sigma)+1+i}$ in $\wp_q(\sigma(\tilde{\alpha} + \rho) - \rho)$ is given by $\binom{r-1-2\ell(\sigma)}{i}$. Thus $\wp_q(\sigma(\tilde{\alpha} + \rho) - \rho) = \sum_{i=0}^{r-1-2\ell(\sigma)} \binom{r-1-2\ell(\sigma)}{i} q^{\ell(\sigma)+1+i} = q^{1+\ell(\sigma)}(1+q)^{r-1-2\ell(\sigma)}$. $\square$

Proposition 3.3. *For* $r \geqslant 1$, $\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k}(1+q)^{r-1-2k} = \sum_{i=1}^{r} q^i$.

4. Non-zero weight spaces

It is fundamental in Lie theory that the zero weight space is a Cartan subalgebra, and that the non-zero weights of $L(\tilde{\alpha})$, the adjoint representation of $\mathfrak{g}$, are the roots and have multiplicity 1. We visit this from our point of view in the case when $\mathfrak{g} = \mathfrak{sl}_{r+1}$. Let $r \geqslant 1$, and $n = r + 1$.

Theorem 4.1. *If* $\mu \in P_+(\mathfrak{sl}_n)$ *and* $\mu \neq 0$, *then* $\mathcal{A}(\tilde{\alpha}, \mu) = \begin{cases} \{1\} \text{ if } \mu = \tilde{\alpha}, \\ \emptyset \text{ otherwise}. \end{cases}$

Recall that given $\mu \in P(\mathfrak{sl}_n)$, there exists $w \in W$ and $\xi \in P_+(\mathfrak{sl}_n)$ such that $w(\xi) = \mu$ and also recall that weight multiplicities are invariant under W (Propositions 3.1.20, 3.2.27 in [2]). Thus it suffices to consider $\mu \in P_+(\mathfrak{sl}_n)$. Setting $q = 1$ in Proposition 3.3 shows $m(\tilde{\alpha}, 0) = r$, while Theorem 4.1 implies $m(\tilde{\alpha}, \tilde{\alpha}) = 1$, and $m(\tilde{\alpha}, \mu) = 0$, whenever $\mu \in P_+(\mathfrak{sl}_n) - \{0, \tilde{\alpha}\}$.

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