C. R. Acad. Sci. Paris, Ser. I 349 (2011) 1053–1058

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial Differential Equations/Mathematical Physics

Characteristic Cauchy problem for the Einstein equations with Vlasov and Scalar matters in arbitrary dimension

Problème de Cauchy caractéristique pour les équations d'Einstein–Vlasov–Champ Scalaire en dimension quelconque

Jean Baptiste Patenou

Department of Mathematics, Faculty of Science, University of Yaounde I, P.O. Box 812, Yaounde, Cameroon

ARTICLE INFO

Article history:
Received 23 June 2011
Accepted after revision 23 August 2011
Available online 25 September 2011

Presented by Yvonne Choquet-Bruhat

ABSTRACT

We derive and solve on two null intersecting smooth hypersurfaces, a set of hierarchical constraints equations, suitable with the proof of an existence theorem for the Einstein equations with Vlasov and Scalar matters, in temporal gauge.

RÉSUMÉ

On établit et résout sur la réunion de deux hypersurfaces caractéristiques régulières sécantes, un système hiérarchisé d'équations des contraintes compatibles avec la preuve d'un théorème d'existence pour les équations d'Einstein–Vlasov–Champ Scalaire en jauge temporelle.

Version française abrégée

Pour $1+n \geqslant 4$, on considère le problème de Cauchy mal posé $\mathcal{P}$ où $\mathcal{Y}$ est le domaine au dessus de l'hypersurface initiale $\mathcal{I} = \mathcal{I}^0 \cup \mathcal{I}^1 (\mathcal{I}^\omega : x^0 + (-1)^\omega x^1 = 0, \omega = 0, 1)$ de $\mathbb{R}^{n+1}$ et l'objectif est de construire un espace-temps (M, g) de source (Φ, ρ) où M est une variété, g une métrique Lorentzienne sur M, Φ est un multiplet scalaire sur M, ρ est la densité des particules sur $\mathbb{P}$; de telle sorte que (g, Φ, ρ) soit solution des équations d'Einstein–Vlasov–Champ Scalaire et que $\mathcal{I}$ soit une hypersurface caractéristique de (M, g). Pour parvenir à notre but, nous considérons pour la métrique les condition de shift nul $g_{0i} = 0$ et d'harmonicité par rapport à x^0 : $\nabla^\nu \nabla_\nu x^0 = \Gamma^0 = 0$ (voir [2]), aboutissant au système du troisième ordre $(H_{\overline{g}}, H_\Phi, H_\rho)$ dit système réduit des équations d'Einstein–Vlasov–Champ Scalaire. Etant donné que si $(\overline{g}, \Phi, \rho)$ est une solution du système réduit $(H_{\overline{g}}, H_\Phi, H_\rho)$, alors le tenseur $\mathfrak{C}_{\mu\nu} = G_{\mu\nu} - T_{\mu\nu}$ correspondant à (g, Φ, ρ) vérifie un système hyperbolique linéaire homogène du second ordre (voir [2]) et est donc nul s'il est nul initialement ; et en vue d'obtenir une solution des équations d'Einstein–Vlasov–Champ Scalaire, nous construisons les données initiales $(\overline{g}_0, k_0, \Phi_0, \rho_0)$ en résolvons sur chaque hypersurface $\mathcal{I}^\omega$, $\omega = 0, 1$ le système $\mathfrak{C}(e_\omega, e_\nu)$. Précisement nous analysons et décrivons le système $\mathfrak{C} = 0$ en repère mobile isotrope particulier (e_α) (voir [5,1]), exhibant ainsi deux sous-systèmes $\mathfrak{C}(e_\omega, e_\nu) = 0$, relatifs à e_0 et e_1 où le système $\mathfrak{C}(e_\omega, e_\nu) = 0$ est celui original hierarchisé des contraintes que nous résolvons sur $\mathcal{I}^\omega$. Nous obtenons ainsi le système complet $(\overline{g}_0, k_0, \Phi_0, \rho_0)$ des données initiales de $\mathcal{P}_0$ satisfaisant les contraintes munies des données indépendantes continues sur $\mathcal{S}$ et de classe $\mathcal{C}^\infty$ sur $\mathcal{I}^\omega$, τ^ω, Θ^ω_{ab}, Φ^ω_0, ρ^ω_0 et certaines données supplémentaires sur $\mathcal{S}$. Le lapse est ensuite

E-mail address: jpatenou@yahoo.fr.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.08.018

défini dans $\mathcal{P}_0$ par $\tau = c(x^i)\sqrt{|\bar{g}|}$ où c, la densité scalaire sur $\Sigma_t : x^0 = t$ est donnée par $c(x^i) = \frac{\tau^\omega}{\sqrt{|\bar{g}_0|}}$ si $(-1)^\omega x^1 \leqslant 0$. Munies des données construites $(\bar{g}_0, k_0, \Phi_0, \rho_0)$ on résout le problème $\mathcal{P}_0$ dans un voisinage de $\mathcal{S}$ par la transformation de ce dernier en un problème de Cauchy ordinaire à données nulles sur l'hypersurface spatiale $\Sigma_0 : x^0 = 0$ (voir [7]), par la combinaison de la théorie de Leray des systèmes hyperboliques et de la méthode des caractéristiques et par application du théorème du point fixe dans un cadre d'espaces fonctionnels plus ou moins classiques (voir [3,4]). Enfin pour la préservation de la jauge nous montrons que sur $\mathcal{I}^\omega$, $\omega = 0, 1$; le système $\mathfrak{C}(e_{\omega'}, e_i) = 0, \omega' \neq \omega$ est équivalent à un système linéaire homogène à données nulles, fortement hyperbolique (voir [8]) en $\mathfrak{C}_{0i}$ sur $\mathcal{I}^\omega$ dès que $\mathfrak{C}(e_\omega, e_\nu) = 0$, $H_{\bar{g}}$ y sont vérifiés et le lapse $\tau \leqslant 2$. On en déduit que $\mathfrak{C}_{\alpha\beta} = 0$ dans un voisinage de $\mathcal{S}$ puisque $\mathfrak{C}_{\alpha\beta} = 0$ sur $\mathcal{I}$ et les $\mathfrak{C}_{\alpha\beta}$ vérifient un système hyperbolique de Leray linéaire homogène (voir [2]). Tous les données et résultats sont $\mathcal{C}^\infty$, ceux en classe de Sobolev et même le cas où les données sont portées par un cône caractéristique sont renvoyés à un travail séparé et ultérieur.

1. The Einstein–Vlasov–Scalar field equations relative to a null adapted frame

Our aim in this work is to provide a space-time (M, g), that is an $(n+1)$-dimensional Lorentzian manifold which has to satisfy the Einstein equations

$$H_g : G_{\mu\nu} \equiv R_{\mu\nu} - 2^{-1} g_{\mu\nu} R = T_{\mu\nu}. \tag{1}$$

Here $R_{\mu\nu}$, R denote the Ricci tensor and scalar curvature of g respectively; $T_{\mu\nu}$ is termed the stress-energy-momentum tensor of matter which here refers to Vlasov and Scalar matters. These equations are coupled to the Vlasov equation giving a statistical description of a collection of particles:

$$H_\rho : p^\alpha \frac{\partial \rho}{\partial x^\alpha} - \gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial \rho}{\partial p^i} = 0 \tag{2}$$

and the wave equations relative to the metric g for the matter fields $\Phi = (\Phi^I)$ of potential V

$$H_\Phi : \Box_g \Phi = \frac{\mathrm{d}V}{\mathrm{d}\Phi}(\Phi); \tag{3}$$

expressing the integrability condition for Einstein equations namely the divergence-free of

$$T_{\alpha\beta} = \partial_\alpha \Phi \partial_\beta \Phi - \frac{1}{2} g_{\alpha\beta}\big(g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi + V(\Phi)\big) - \int_{\mathbb{R}^n} \frac{\rho(x^\nu, p^\mu) p_\alpha p_\beta \sqrt{\det g}}{p^0}\, \mathrm{d}p. \tag{4}$$

n is an arbitrary natural number $\geqslant 3$, (x^α) refers to a set of coordinates in the looked-for space-time, the p^i stand as the spatial components of the momentum of the particles of rest mass m and density $\rho \equiv \rho(x, p^0, p^i)$ moving towards the future $(p^0 > 0)$ in the hypersurface $\mathbb{P}$ of the tangent bundle TM named the mass shell, $\mathbb{P} := \{(x, p^0, p^i)/g_{\mu\nu} p^\mu p^\nu = -m^2\}$, $\mathrm{d}p = \mathrm{d}p^1 \ldots \mathrm{d}p^n$. To achieve our goal we require that in a global set of coordinates x^α of $\mathbb{R}^{n+1}$, the metric we are looking for has a zero shift and that x^0 is harmonic with respect to g (see [2]),

$$g = -\tau^2 \big(\mathrm{d}x^0\big)^2 + \bar{g}_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j, \quad \tau = c\big(x^i\big)\sqrt{\det \bar{g}}; \tag{5}$$

and associate to the ill-posed Cauchy problem $\mathcal{P}$ and Cauchy problem $\mathcal{P}_0$ in $\mathcal{Y} = \{(x^\mu)/x^0 \geqslant |x^1|\}$ and on $\mathcal{I} = \mathcal{I}^0 \cup \mathcal{I}^1 (\mathcal{I}^\omega : x^0 + (-1)^\omega x^1 = 0)$ for the reduced third order system of unknown $(\bar{g}, \Phi, \rho)$ (see [2]).

$$\mathcal{P}_0 \begin{cases} (H_{\bar{g}}, H_\Phi, H_\rho) & \text{in } \mathcal{Y}, \\ (\bar{g}, \partial_0 \bar{g}, \Phi, \rho) = (\bar{g}_0, k_0, \Phi_0, \rho_0) & \text{on } \mathcal{I}, \end{cases} \qquad \mathcal{P} \begin{cases} (H_g, H_\Phi, H_\rho) & \text{in } \mathcal{Y}, \\ (g, \partial_0 g, \Phi, \rho) = (g_0, h_0, \Phi_0, \rho_0) & \text{on } \mathcal{I}, \end{cases} \tag{6}$$

$$H_{\bar{g}} \equiv \partial_0 R_{ij} - \overline{\nabla}_i R_{j0} - \overline{\nabla}_j R_{i0} = \partial_0 \Lambda_{ij} - \overline{\nabla}_i \Lambda_{j0} - \overline{\nabla}_j \Lambda_{i0}; \qquad \Lambda_{\mu\nu} = T_{\mu\nu} + 2^{-1} g_{\mu\nu} R \quad (\overline{\nabla} \equiv \text{connection w.r.t. } \bar{g}). \tag{7}$$

As the tensor $\mathfrak{C}_{\mu\nu} = G_{\mu\nu} - T_{\mu\nu}$ corresponding to (g, Φ, ρ) where $(\bar{g}, \Phi, \rho)$ satisfies $\mathcal{P}_0$ verifies a second order Leray hyperbolic linear homogeneous system (see [2]), in order to have a solution for the Einstein–Vlasov–Scalar field equations (H_g, H_Φ, H_ρ), we construct the data $(\bar{g}_0, k_0, \Phi_0, \rho_0)$ solving the constraints $\mathfrak{C}(e_\omega, e_\nu) = 0$, $\omega = 0, 1$ on $\mathcal{I}^\omega$. Namely, we assume that our searched space-time admits a double null foliation $C_-(\lambda)$ and $C_+(\nu)$ by the null-level surfaces of two functions w and $\underline{w}$ solutions of the eikonal equation with data $w_{/\mathcal{I}^0} = -2x^1$, $\underline{w}_{/\mathcal{I}^1} = 2x^1$, the Einstein equations split in two groups $\mathfrak{C}(e_\omega, e_\nu) = 0$, $\omega = 0, 1$ w.r.t. a null pair $\{e_0, e_1\}$ subset of the null moving frame e_α adapted to this foliation (see [5,1]). The equations corresponding to e_0 read

$$-e_0(\operatorname{tr}\chi) + 2\omega \operatorname{tr}\chi - |\chi|^2_\gamma = \big(e_0(\Phi)\big)^2 - \int_{\mathbb{R}^n} \tau^2 \rho\big(x^\nu, p^\delta\big) \frac{(p^0 + p^j \partial_j w)^2}{p^0} \sqrt{\det g}\, \mathrm{d}p, \tag{8}$$

$$-e_0(\xi_A) + r_A(\xi, .) = e_0(\Phi) e_A(\Phi) + \int_{\mathbb{R}^n} \tau \rho\big(x^\nu, p^\delta\big) e^i_A p_i \frac{p^0 + p^j \partial_j w}{p^0} \sqrt{\det g}\, \mathrm{d}p, \tag{9}$$

$$e_0(\widehat{\underline{\chi}}_{AB}) + r_{AB}(\widehat{\underline{\chi}},.) = e_A(\Phi)e_B(\Phi) - \frac{1}{2}\delta_{AB}\big(g^{\gamma\delta}\partial_\gamma\Phi\partial_\delta\Phi + V(\Phi)\big) - \frac{1}{n-1}\delta_{AB}\sum_C T(e_C,e_C) - \int_{\mathbb{R}^n}\rho(x^\nu,p^\delta)\frac{e_A^i e_B^j p_i p_j}{p^0}\sqrt{\det g}\,\mathrm{d}p, \tag{10}$$

$$e_0(\operatorname{tr}\underline{\chi}) + 2\omega\operatorname{tr}\underline{\chi} + \frac{2}{n-1}\operatorname{tr}\chi\operatorname{tr}\underline{\chi} - (n-1)\big(|\underline{\eta}|^2 + \underline{\operatorname{div}}\,\underline{\eta}\big) + (n-1)(n-2)\mathbf{K}_0 = \sum_A T(e_A,e_A) - T, \tag{11}$$

$$-2e_0(\underline{\omega}) + r_0(\underline{\omega},.) = e_0(\Phi)e_1(\Phi) + g^{\gamma\delta}\partial_\gamma\Phi\partial_\delta\Phi + V(\Phi) - \int_{\mathbb{R}^n}\tau^2\rho(x^\nu,p^\delta)\frac{(p^0)^2 + p^0(p^j\partial_j w + p^j\partial_j\underline{w}) + (p^j\partial_j w)(p^j\partial_j\underline{w})}{p^0}\sqrt{\det g}\,\mathrm{d}p, \tag{12}$$

$$T = g^{\mu\nu}T_{\mu\nu} = \frac{1-n}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{n+1}{2}V(\Phi) + m^2\int_{\mathbb{R}^n}\frac{\rho(x^\nu,p^\delta)}{p^0}\sqrt{\det g}\,\mathrm{d}p.$$

The tensors χ, $\underline{\chi}$, ξ are the two null second fundamental forms and the torsion of the $(n-1)$-dimensional surfaces $C_{\lambda\mu} \equiv C_-(\lambda)\cap C_+(\nu)$, $\omega = \frac{1}{4}g(\nabla_{e_0}e_0,e_1)$, $\underline{\omega} = \frac{1}{4}g(\nabla_{e_1}e_1,e_0)$, $\eta(X) = -\frac{1}{2}g(\nabla_{e_1}X,e_0)$, $\underline{\eta}(X) = -\frac{1}{2}g(\nabla_{e_0}X,e_1)$ for X tangent to $C_{\lambda\mu}$ and ∇ the connection of the space-time. The vectors fields e_α are defined as $e_0^\nu = -\tau g^{\nu\mu}\partial_\mu w$, $e_1^\nu = -\tau g^{\nu\mu}\partial_\mu\underline{w}$, and (e_A), $A = 2,\dots,n$ is an orthonormal frame on $C_{\lambda\mu}$. $\underline{\nabla}$, $\mathbf{K}_0$ are the connection and Gauss curvature of $C_{\lambda\mu}$. The details of the terms not written explicitly and equations relative to e_1 are in [6]; see also [5] for analogous arguments for the Null decomposition of Einstein equations in Vacuum.

2. The construction of the characteristic initial data

By restricting the previous equations (8)–(12) on the initial hypersurface $\mathcal{I}$ one derives the following theorem:

Theorem 2.1. *Let be considered $\mathcal{I}$ the union of the two hypersurfaces $\mathcal{I}^\omega : \vartheta^{\omega+1} \equiv x^0 + (-1)^\omega x^1 = 0$, $\omega = 0,1$ of $\mathbb{R}^{n+1}$ endowed with a global set of coordinates x^μ and denote $\mathcal{S} = \mathcal{I}^0\cap\mathcal{I}^1$; let be given on $\mathcal{I}$ a non-negative smooth function τ, $(\tau(0,x^a) = 1)$, a smooth scalar multiplet Φ_0, and smooth functions Θ_{ab}, $a,b = 2,\dots,n$ that make up a symmetric positive definite matrix, on $\mathcal{I}\times\mathbb{R}^n$, a non-negative smooth function ρ_0 with compact support such that $\operatorname{supp}(\rho_0)\cap\mathcal{S} = \emptyset$ (rest mass $m > 0$) or $\operatorname{supp}(\rho_0)\cap(\mathcal{S}\times\{(p^i)/\sum_a(p^a)^2 = 0\}) = \emptyset$ (zero rest mass); and on $\mathcal{S}$, $\mathcal{C}^\infty$ functions Ψ^0, Ψ^1 and υ_a, $a = 2,\dots,n$. Then there exist on every $\mathcal{I}^\omega$ smooth functions Λ, g_0, k_0, and correspondingly smooth functions Λ, g_0, k_0, Φ_0, ρ_0 solving the constraints for the Einstein–Vlasov–Scalar field equations and such that $g_{0_{00}} = -\tau^2$, $g_{0_{ab}} = \Lambda^2\Theta_{ab}$, $\mathcal{I}$ is null w.r.t. g_0, the initial rate of change of the area element of the $(n-1)$-dimensional leaves of $\mathcal{I}$ under displacement along $\mathcal{I}^0$ and $\mathcal{I}^1$ are respectively Ψ^0, Ψ^1, and the tensor $\upsilon = (\upsilon_a)$ is the torsion of the $(n-1)$-dimensional leave $\mathcal{S}$.*

2.1. The strategy of the proof of Theorem 2.1

We treat the data on $\mathcal{I}^0$ and the process is similar on $\mathcal{I}^1$. The assumption that $\mathcal{I}^0$ is null w.r.t. to the metric of the form (6) is equivalent to $g^{11} = \tau^{-2}$. In coordinates x^ν the trace on $\mathcal{I}^0$ of the metric reduces to

$$g_0 = -\tau^2\big(\mathrm{d}x^0\big)^2 + \tau^2\big(\mathrm{d}x^1\big)^2 + g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b. \tag{13}$$

We emphasize that this is not a restriction on the trace of the metric but rather a consequence of our gauge conditions and the nature of the initial surface. Indeed the covariant vector $n := \operatorname{grad}\vartheta^1$ is a null vector normal and tangent to $\mathcal{I}^0$ with components $n_0 = 1$, $n_1 = 1$, $n_a = 0$. The null geodesics generating $\mathcal{I}^0$ have tangent vector l with components $l^0 = \tau^{-2}$, $l^1 = -\tau^{-2}$, $l^a = -g^{1a}$. By uniqueness of null directions tangent to $\mathcal{I}^0$, it follows that $g^{1a} = 0$. On the other hand the null condition for $\mathcal{I}^0$ and the algebraic relations between the coefficients of the metric and its inverse imply that $g_{1a} = -\tau^2 g_{ab}g^{1b}$, $g_{11} = \tau^2(1 + \tau^2 g_{ab}g^{1a}g^{1b})$, one deduces from $g^{1a} = 0$ that $g_{1a} = 0$, $g_{11} = \tau^2$. Let denote ϑ, $\underline{\vartheta}$ the affine parameters on $\mathcal{I}^0$ and $\mathcal{I}^1$ respectively, then $\frac{\partial}{\partial\vartheta} := -\tau^{-2}\frac{\partial}{\partial x^1}$ and $\frac{\partial}{\partial\underline{\vartheta}} := \tau^{-2}\frac{\partial}{\partial x^1}$. The light rays of $\mathcal{I}^0$ have equations $\vartheta^1 = 0$, $x^a = cste^a = \vartheta^a$ where the ϑ^a are local coordinates on $\mathcal{S}$. We define a foliation of $\mathcal{I}^0$ by the $(n-1)$-dimensional surfaces $\mathcal{I}^0_\nu = \{p\in\mathcal{I}^0/\vartheta(p) = \nu\}$. The induced metric γ on $\mathcal{I}^0_\nu$, $\gamma_{ab} \equiv (g_{0/\mathcal{I}^0_\nu})_{ab}$ is supposed conformal to Θ_{ab}, that is $\gamma_{ab} = \Lambda^2\Theta_{ab}$. From the definition of null second fundamental form $\chi(2\chi = \mathcal{L}_l\gamma)$ one infers the following expressions on $\mathcal{I}^0_\nu$.

$$\chi_{ab} = \Lambda^2\left(\Theta_{ab}\partial_\vartheta\ln\Lambda + \frac{1}{2}\partial_\vartheta\Theta_{ab}\right), \qquad \operatorname{tr}\chi = \gamma^{ab}\chi_{ab} = \partial_\vartheta(\ln\sqrt{\det\gamma}) = \partial_\vartheta\big(\ln\big(\Lambda^{n-1}\sqrt{\det\Theta}\big)\big), \tag{14}$$

$$|\chi|^2 = \frac{1}{n-1}\big(\partial_\vartheta\big(\ln\Lambda^{n-1}\sqrt{\det\Theta}\big)\big)^2 - \frac{1}{n-1}\big(\partial_\vartheta(\ln\sqrt{\det\Theta})\big)^2 + \frac{1}{4}\Theta^{ad}\partial_\vartheta\Theta_{ab}\Theta^{cb}\partial_\vartheta\Theta_{cd}. \tag{15}$$

As the right hand of the first Eq. (8) of our hierarchy is harmful containing all the coefficients of the metric, we invoke our gauge conditions. Eq. (8) is rewritten using Eqs. (14)–(15) and $\omega = -2^{-1}\partial_\vartheta\tau$ as:

$$\tau\partial^2_\vartheta\big(\ln\Lambda^{n-1}\sqrt{\det\Theta}\big)+\partial_\vartheta\tau\partial_\vartheta\big(\ln\Lambda^{n-1}\sqrt{\det\Theta}\big)+\frac{1}{n-1}\big(\partial_\vartheta\big(\ln\Lambda^{n-1}\sqrt{\det\Theta}\big)\big)^2$$
$$-\frac{1}{n-1}\big(\partial_\vartheta(\ln\sqrt{\det\Theta})\big)^2+\frac{1}{4}\Theta^{ad}\partial_\vartheta\Theta_{ab}\Theta^{cb}\partial_\vartheta\Theta_{cd}=-\tau^2[\partial_\vartheta\Phi]^2+\int_{\mathbb{R}^n}\tau^4\rho_0\big(x^i,p^j\big)\frac{(p^0-p^1)^2}{p^0}\,\mathrm{d}p. \tag{16}$$

Using

$$p^0=\tau^{-1}\sqrt{m^2+\tau^2\big(p^1\big)^2+\Lambda^2\Theta_{ab}p^ap^b},\qquad \Lambda=(\sqrt{\det\Theta})^{1-n}\exp\left[\frac{1}{n-1}\left(\Psi_0+\int_0^\vartheta\Psi\big(\vartheta',\vartheta^a\big)\,\mathrm{d}\vartheta'\right)\right]$$

Eq. (16) leads to the following Cauchy problem where $\Psi=\partial_\vartheta\ln\Lambda^{n-1}\sqrt{\det\Theta}$:

$$\partial_\vartheta\Psi+\partial_\vartheta(\ln\tau)\Psi+\frac{\tau^{-1}}{n-1}\Psi^2+L\big(x^1,\vartheta^a,\tau,\Theta,\Psi,\Phi_0,\rho_0\big)=0,\qquad \Psi_{|\mathcal{S}}=\Psi_0. \tag{17}$$

From this Cauchy problem (17) one has on $(\mathcal{I}^0)\subset\mathcal{I}^0$ the conformal factor, and then g_{0ab}. For the next step we determine the vectors fields e_A on a domain $(\mathcal{I}'^0)\subset\mathcal{I}^0$ Fermi-transported [2,5] thanks to the following non-linear Cauchy problem compatible with the structure of our searched-for space-time.

$$\partial_\vartheta e^c_A+\tau^{-1}\sum_C\chi_{ab}e^a_Ae^b_Ce^c_C=0,\qquad e^c_{A|\mathcal{S}}=\delta^c_A.$$

To pursue with the construction of the data, we pose and solve globally on $(\mathcal{I}'^0)$ a Cauchy problem for the following system with initial data $\xi_{A|\mathcal{S}}=\upsilon_A$ corresponding to the restriction of (9) to $\mathcal{I}^0$ constructing ξ_A:

$$-\tau\partial_\vartheta\xi_A-\frac{1}{n-1}\partial_\vartheta\big[\ln\big(\Lambda^{n-1}\sqrt{\det\Theta}\big)\big]\xi_A+Q_A=\tau\partial_\vartheta(\Phi)e_A(\Phi)+\int_{\mathbb{R}^n}\tau^3\rho_0\big(x^i,p^j\big)e^c_Ap_c\frac{p^0-p^1}{p^0}\,\mathrm{d}p\quad\text{on }\big(\mathcal{I}^0\big).$$

To continue our process, we determine the restriction on $\mathcal{I}^0$ of the first derivatives of the component Φ of any solution of $\mathcal{P}_0$. This is obtained by restricting to $\mathcal{I}^0$ the wave equations $\Box\Phi=V'(\Phi)$ (3) which give rise to transport equations along the null generators of $\mathcal{I}^0$, we therefore solve in $(\mathcal{I}'^0)$ the following Cauchy problem

$$2\frac{\mathrm{d}}{\mathrm{d}\vartheta}\big[\partial_0\Phi^I\big]+g^{ij}\frac{\partial^2\Phi_0}{\partial x^i\partial x^j}+F^I\big(x^j,\Phi_0,\big[\partial_0\Phi^K\big]\big)=0,\qquad \big[\partial_0\Phi^I\big]_{/\mathcal{S}}=\frac{1}{2}(\partial_1\underline{\Phi}_0-\partial_1\Phi_0),\qquad \underline{\Phi}_0=\Phi_{/\mathcal{I}^1}.$$

The next step consists in determining $\operatorname{tr}\underline{\chi}$ and this is obtained by solving globally on $(\mathcal{I}''^0)$ the following Cauchy problem associated to the system (11) with initial datum $\operatorname{tr}\underline{\chi}_{/\mathcal{S}}=\underline{\Psi}_0$.

$$\tau\partial_\vartheta\operatorname{tr}\underline{\chi}+\left(-\frac{1}{4}\partial_\vartheta\ln\tau+\frac{2}{n-1}\partial_\vartheta\big[\ln\big(\Lambda^{n-1}\sqrt{\det\Theta}\big)\big]\right)\operatorname{tr}\underline{\chi}+S=\sum_A T(e_A,e_A)-T,\qquad \operatorname{tr}\underline{\chi}_{/\mathcal{S}}=\underline{\Psi}_0.$$

At this level we construct $\widehat{\underline{\chi}}$ from (10) by solving globally on $(\mathcal{I}''^0)$ the following Cauchy problem.

$$\begin{cases}\tau\partial_\vartheta\widehat{\underline{\chi}}_{AB}+\left(-\frac{1}{2}\partial_\vartheta\tau+\frac{1}{n-1}\partial_\vartheta\big[\ln\big(\Lambda^{n-1}\sqrt{\det\Theta}\big)\big]+A_\chi\right)\widehat{\underline{\chi}}_{AB}+S_{AB}\\ \quad=e_A(\Phi)e_B(\Phi)-\frac{1}{2}\delta_{AB}\big\{g^{ij}\partial_i[\Phi]\partial_j[\Phi]+2g^{1i}\partial_i[\Phi][\partial_0\Phi]+V(\Phi)\big\}\\ \quad\quad-\frac{1}{n-1}\delta_{AB}\sum_C T(e_C,e_C)-\int_{\mathbb{R}^n}\frac{\tau^2e^c_Ae^d_Bp_cp_d}{p^0}\rho_0\big(x^i,p^j\big)\,\mathrm{d}p\quad\text{on }\big(\mathcal{I}'^0\big),\\ \widehat{\underline{\chi}}_{AB/\mathcal{S}}=(\det\Theta)^{1-n}_{|\mathcal{S}}\exp\left[\frac{2\underline{\Psi}_0}{n-1}\right]\left(2\Theta_{ab}\partial_{\underline{\vartheta}}\ln\Lambda+\frac{1}{2}\partial_{\underline{\vartheta}}\Theta_{ab}+\frac{1}{n-1}\Theta_{ab}\partial_{\underline{\vartheta}}\ln\sqrt{\det\Theta}\right)_{|\mathcal{S}}.\end{cases}$$

The term $\partial_{\underline{\vartheta}}\ln\Lambda$ is given on $\mathcal{S}$ by the expression of Λ on $\mathcal{I}^1$, $\Lambda=(\sqrt{\det\Theta})^{1-n}\exp[\frac{1}{n-1}(\underline{\Psi}_0+\int_0^{\underline{\vartheta}}\underline{\Psi}(\underline{\vartheta}',\vartheta^a)\,\mathrm{d}\underline{\vartheta}')]$.

To continue the process, we precise the values of q_i, q'_i on $\mathcal{I}^0$ as $q_i=[\partial_iw]=-\delta^1_i$ and $q'_i=\partial_i\underline{w}=\delta^1_i$ and solve globally the following Cauchy problem associated to (12)

$$\begin{cases}-4\tau\partial_\vartheta\underline{\omega}+(\partial_\vartheta\tau)\underline{\omega}+Q=\tau^2\partial_\vartheta\Phi\partial_{\underline{\vartheta}}\Phi+g^{ij}\partial_i[\Phi]\partial_j[\Phi]+2g^{1i}\partial_i[\Phi][\partial_0\Phi]\\ \quad+V(\Phi)-\int_{\mathbb{R}^n}\tau^4\frac{(p^0)^2-(p^1)^2}{p^0}\rho_0\big(x^i,p^j\big)\,\mathrm{d}p\quad\text{on }\big(\mathcal{I}'^0\big),\ \underline{\omega}_{/\mathcal{S}}=-\frac{\partial\underline{\tau}}{\partial\underline{\vartheta}}\big(0,x^a\big).\end{cases}$$

To conclude with the construction of the data on $\mathcal{I}^0$ we use the algebraic relations between $\chi, \underline{\chi}, \xi \ldots$ and $g_{\mu\nu}, \partial_0 g_{\mu\nu} \ldots$ to determine the coefficients $[\partial_0 g_{\mu\nu}], (\mu,\nu) \neq (1,1)$ and the gauge constraint $\Gamma^0 = 0$ to determine the coefficient $[\partial_0 g_{11}]$, one deduces k_0.

$$\begin{cases} \underline{\chi}_{AB} + \chi_{AB} = e^c_A e^d_B e^0_0 \partial_0 g_{cd}, \quad 4\xi_A = -2e^0_0 e^c_A e^1_0 \partial_0 g_{1c} - 2e^d_0 e^B_d (\underline{\chi}_{AB} + \chi_{AB}), \\ \underline{\omega} + \omega = (2\tau)^{-3} \partial_0 g_{00}, \quad \partial_0 g_{11} = \tau^2 \big(2\partial_0 \ln \sqrt{-g_{00}} - g^{ab} \partial_0 g_{ab}\big). \end{cases}$$

3. The existence theorem for the Einstein–Vlasov–Scalar field equations

With respect to the gauge constraint $\nabla^\mu \nabla_\mu x^0 = 0$, one sets in $\mathcal{P}_0$ the lapse as $\tau = c(x^i)\sqrt{|\overline{g}|}$ where c, the scalar density on $\Sigma_t : x^0 = t$, is given by the constructed data by $c(x^i) = \frac{\tau^\omega}{\sqrt{|\overline{g}_0|}}$ if $(-1)^\omega x^1 \leqslant 0$.

Theorem 3.1. *Let be considered $\mathcal{I}$ the union of the two hypersurfaces $\mathcal{I}^\omega : x^0 + (-1)^\omega x^1 = 0$, $\omega = 0, 1$ of $\mathbb{R}^{n+1}$ endowed with a global set of coordinates x^μ and denote $\mathcal{S} = \mathcal{I}^0 \cap \mathcal{I}^1$; let be given on $\mathcal{I}$ a non-negative smooth function $\tau \leqslant 2$, $(\tau(0, x^a) = 1)$, a smooth scalar multiplet Φ_0, and smooth functions Θ_{ab}, $a, b = 2, \ldots, n$ that make up a symmetric positive definite matrix, on $\mathcal{I} \times \mathbb{R}^n$, a non-negative smooth function ρ_0 with compact support such that $\operatorname{supp}(\psi) \cap \mathcal{S} = \emptyset$ (rest mass $m > 0$) or $\operatorname{supp}(\psi) \cap (\mathcal{S} \times \{(p^i)/\sum_a (p^a)^2 = 0\}) = \emptyset$ (zero rest mass); and on $\mathcal{S}$, $\mathcal{C}^\infty$ functions $\Psi^0, \underline{\Psi}^0$ and υ_a, $a = 2, \ldots, n$. Then there exists an open neighborhood M of $\mathcal{I}^0 \cap \mathcal{I}^1$ in $\mathcal{Y}$, a unique smooth non-negative function Λ on $\mathcal{I} \cap M$, unique $\mathcal{C}^\infty$ Lorentz metric g and scalar multiplet Φ on M, and a unique $\mathcal{C}^\infty$ non-negative function $\rho(x^\nu, p^\alpha)$ in $M \times \mathbb{P}_x$; $\mathbb{P} := \{(x, p)/g_{\mu\nu} p^\mu p^\nu = -m^2,\ p^0 > 0\}$ such that the triplet (g, Φ, ρ) satisfies the Einstein–Vlasov–Scalar field equations, $g_{00} = -\tau^2$, $g_{ab} = \Lambda^2 \Theta_{ab}$ on $\mathcal{I}$, the initial rate of change of the area element of $(n-1)$-dimensional leaves of $\mathcal{I}$ under displacement along $\mathcal{I}^0$ and $\mathcal{I}^1$ are respectively Ψ^0 and Ψ^1, and υ is the torsion of $\mathcal{S}$. Furthermore M admits a double null foliation by the level surfaces of the functions w and $\underline{w}$ solutions of the eikonal equation $g^{\mu\nu}\partial_\mu u \partial_\nu u = 0$ with initial data $u_{/\mathcal{I}^0} = -2x^1$ and $u_{/\mathcal{I}^1} = 2x^1$.*

The strategy of the proof of Theorem 3.1. One uses a method analogous to the one of [7] for the reduction of the characteristic initial value problem $\mathcal{P}_0$ to a Cauchy problem on $\Sigma_0 : x^0 = 0$, and combine the Leray theory of hyperbolic systems and the method of characteristics for the linearized equations, paying attention that the solution of the linearized Vlasov equation has its support in $\mathcal{Y} \times \mathbb{P}_x$. Indeed the property $|p^0| \geqslant |p^1|$ is an essential argument. Concerning the procedure of reduction of the characteristic problem $\mathcal{P}_0$ to a Cauchy problem, let emphasize that differentiating the equations of $\mathcal{P}_0$ and restricting to $\mathcal{P}_0$ gives rise to a collection of functions which agree with the derivatives of any solution of $\mathcal{P}_0$ provided that $|p^0| > |p^1|$. The validity of such condition is as follows. Let λ be a real number, then $g_{ij}(\lambda p^i + g^{1i})(\lambda p^j + g^{1j}) \geqslant 0$; as our initial surfaces are null w.r.t. the metric $(g^{11} = \tau^{-2})$, this inequality leads to $(p^0)^2 \geqslant m^2\tau^{-2} + (p^1)^2$. One has $|p^0| \geqslant |p^1|$ in general and $|p^0| > |p^1|$ in the case of non-zero rest mass. For the case of zero rest mass, we use a support condition, indeed $\sum_a (p^a)^2 > 0$ implies that $|p^0| > |p^1|$. One concludes with the proof of the existence theorem for $\mathcal{P}^0$ by establishing energy estimates and applying the fixed point theorem in functional spaces more or less classical (see [3,4]). For the gauge preservation we prove that the subclass $\mathfrak{C}(e_{\omega'}, e_i) = 0$, $\omega' \neq \omega$ of $\mathfrak{C}(e_\mu, e_\nu) = 0$ or equivalently $\mathfrak{C}_{0i} = 0$ is, from now on, verified on $\mathcal{I}^\omega$. This is achieved by combining the restriction of $H_{\overline{g}}$ on $\mathcal{I}^\omega$ and the solved constraints $\mathfrak{C}(e_\omega, e_\nu) = 0$; one deduces that the $\mathfrak{C}_{0i}$ satisfy on $\mathcal{I}^\omega$ a linear homogeneous system of the following form with zero data

$$\begin{aligned} &A^{jl}_k(x')\partial_j[\mathfrak{C}_{0l}] + f_k\big(x', [\mathfrak{C}_{0i}]\big) = 0, \\ &A^{jl}_k(x') = \epsilon q^j \delta^l_k + \epsilon' g^{jl} q_k + q^l \delta^j_k + 64\tau^{-2} q_k q^j q^l, \quad \epsilon = 1 + 4\tau^{-2}, \\ &\quad \epsilon' = -1 + 4\tau^{-2}, \quad a_{\epsilon\epsilon'} = (\epsilon + \epsilon')\big(1 + \epsilon + \epsilon^2 + \epsilon\epsilon'\big). \end{aligned} \tag{18}$$

The strong hyperbolicity (see [8]) of (18) resides on the fact that $A^{jl}_k q_j$ is invertible as its determinant is $\Delta = \epsilon^{n-1}\tau^{-2n}(1 + \epsilon + \epsilon' + 64\tau^{-4}) > 0$, and $\operatorname{Span}\{\bigcup_{\lambda \in \mathbb{R}} \operatorname{Kernel}[A^{jl}_k(\lambda q_j + \omega_j)]\}$ is of dimension $n-1$, for each co-vector ω_i not proportional to q_i, provided that $\tau \leqslant 2$. The eigenvalues of $(A^{jl}_k q_j)^{-1}.A^{jl}_k \omega_j$ are $\lambda = \tau^2 q^i \omega_i \pm \tau^2 (a_{\epsilon\epsilon'} - \epsilon')^{-1} \sqrt{\tau^{-2}\epsilon'(\epsilon' - a_{\epsilon\epsilon'})[(q^i\omega_i)^2 - \tau^{-2}\omega^i\omega_i]}$ and an $n-2$ multiple eigenvalue $\lambda = \tau^2 q^i \omega_i$. Now as $\mathfrak{C} = 0$ on $\mathcal{I}$, we apply the Leray hyperbolic existence theorem for the homogeneous system in $\mathfrak{C}_{\mu\nu}$ (see [2]) with zero data. As our data and results are smooth, we intend to treat the case of Sobolev classes as well as characteristic cone data in a separate and subsequent work.

Acknowledgement

I am indebted to Professor Marcel Dossa for suggesting this subject and for many improvements in this work.

References

[1] G. Caciotta, F. Nicolò, Global characteristic problem for Einstein Vacuum equations with small initial data (I): The initial data constraints, JHDE 2 (1) (2005) 201–277.

[2] F. Cagnac, Y. Choquet-Bruhat, N. Noutchegueme, in: U. Bruzzo, R. Cianci, E. Massa (Eds.), General Relativity and Gravitational Physics, World Scientific, Singapore, 1987.
[3] Y. Choquet-Bruhat, Problème de Cauchy pour le système intégro différentiel d'Einstein–Liouville, Annales de l'Institut Fourier 21 (3) (1971) 181–201.
[4] Y. Choquet-Bruhat, Norbert Noutchegueme, Système de Yang–Mills–Vlasov en jauge temporelle, Annales de l'Institut Henri Poincaré 55 (3) (1991) 759–787.
[5] S. Klainerman, F. Nicolò, The Evolution Problem in General Relativity, Progress in Mathematical Physics, Birkhäuser, 2003.
[6] J.B. Patenou, Doctorat/PhD thesis, University of Yaounde I (Cameroon), in preparation.
[7] A.D. Rendall, Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations, Proc. Roy. Soc. Lond. A 427 (1990) 221–239.
[8] Oscar Reula, Strongly hyperbolic systems in general relativity, JHDE 1 (2) (2004) 251–269.