C. R. Acad. Sci. Paris, Ser. I 349 (2011) 1083–1087

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Algebraic Geometry

Arcs and wedges on rational surface singularities

Arcs et coins sur une singularité rationnelle de surface

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ARTICLE INFO

Article history:
Received 17 May 2011
Accepted after revision 30 August 2011
Available online 28 September 2011

Presented by Claire Voisin

ABSTRACT

Let (S, P_0) be a rational surface singularity over an algebraically closed field k of characteristic 0, let ν_α be an essential divisorial valuation over (S, P_0), and P_α the stable point of the space of arcs S_∞ corresponding to ν_α. We prove that any wedge centered at P_α lifts to the minimal desingularization. This proves the Nash problem for rational surface singularities, and reduces the Nash problem for surfaces to quasirational normal singularities which are not rational. In positive characteristic, we give a counterexample to the k-wedge lifting problem for a surface for which the Nash map is bijective.

RÉSUMÉ

Soit (S, P_0) une singularité rationnelle de surface sur un corps algébriquement clos k de caractéristique 0, soit ν_α une valuation divisorielle essentielle sur (S, P_0), et P_α le point stable de l'espace des arcs S_∞ qui correspond à ν_α. On démontre que tout coin centré en P_α se relève à la désingularisation minimale. Cela démontre le problème de Nash pour les singularités rationnelles de surface, et réduit le problème de Nash pour les surfaces aux singularités quasi-rationnelles qui ne sont pas rationnelles. En caractéristique positive, on donne un contre-exemple au problème de relèvement de k-coins pour une surface dont l'application de Nash est bijective.

Version française abrégée

Soit k un corps algébriquement clos. Soit X une surface sur k et X_∞ le k-schéma des arcs sur X. Soit $\pi : Y \to X$ la désingularisation minimale de X, $\{E_\alpha\}_{\alpha\in\Lambda}$ les courbes exceptionnelles pour π et $\{\nu_\alpha\}_{\alpha\in\Lambda}$ les valuations divisorielles correspondantes, qu'on appelle *diviseurs essentiels*.

Pour chaque $\alpha \in \Lambda$, soit N_α l'adhérence de Zariski de l'image par $\pi_\infty : Y_\infty \to X_\infty$ de l'ensemble des arcs sur Y centrés en un point de E_α. L'ensemble N_α est un fermé irreductible de X_∞, son point générique P_α est appelé *point stable de* X_∞ défini par ν_α. Étant donnée $Q_0 \in E_\alpha$, on denote $N^\dagger(Q_0)$ l'image par π_∞ de l'ensemble des points Q de Y_∞ dont l'arc correspondant intersecte transversalement E_α en Q_0. On dit que $P \in X_\infty$ est un *élément général de* N_α s'il existe $Q_0 \in E_\alpha \setminus \bigcup_{\alpha'\neq\alpha} E_{\alpha'}$ tel que $P \in N^\dagger(Q_0)$. Par exemple, P_α est un élément général de E_α car $P_\alpha \in N^\dagger(Q_0)$ où Q_0 est le point générique de E_α.

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.08.022

Étant donnée une extension de corps K de k, un K-coin sur X est un k-morphisme $\Phi : \operatorname{Spec} K[[\xi, t]] \to X$. Il determine un k-morphisme $\varphi : \operatorname{Spec} K[[\xi]] \to X_\infty$. L'image par φ du point fermé (resp. générique) est appellée *l'arc spécial* (resp. *générique*) de Φ. On dit que Φ est centré en son arc spécial.

On dit que π *satisfait la propriété de relèvement de k-coins par rapport à* E_α si l'ensemble des points $Q_0 \in E_\alpha$ tels que : "tout k-coin centré en un arc de $N^\dagger(Q_0)$ se relève à Y" est très dense. On dit que π *satisfait la propriété de relèvement de K-coins centrés en* P_α si tout K-coin centré en P_α se relève à Y. Si k est non dénombrable, alors la propriété de relèvement de k-coins par rapport à E_α entraîne la propriété de relèvement de K-coins centrés en P_α [6, prop. 2.9].

Soit P_0 un point fermé de X et soit (S, P_0) le voisinage formel de P_0 sur X. On identifie arcs sur X centrés en P_0 avec arcs sur (S, P_0).

Le résultat principal de cette Note est :

Théorème 0.1. *Soit ν_α un diviseur essentiel pour une singularité rationnelle de surface (S, P_0) sur un corps algébriquement clos de caractéristique* 0. *Soit $\Phi : \operatorname{Spec} K[[\xi, t]] \to S$ un K-coin centré en un élément général de N_α. Alors Φ se relève à la désingularisation minimale de (S, P_0).*

Pour la preuve, une idée fondamentale est de généraliser ce qu'on avait fait pour les singularités sandwich des surfaces dans [5]. Pour cela, on définit des vecteurs caractéristiques $\{\omega_\alpha\}_{\alpha \in \Lambda}$ tels que la donnée du graphe dual Γ de π et des $\{\omega_\alpha\}_{\alpha \in \Lambda}$ contient la même information que la matrice d'intersection des E_α. Pour chaque extrémité ϵ_i de Γ, on prend $x_i \in \mathcal{O}_{S,P_0}$ dont la transformée stricte sur Y intersecte seulement E_{ϵ_i}, et on étudie les valeurs $(\nu(x_i))_i$ pour toute valuation divisorielle ν sur (S, P_0). On en déduit une obstruction combinatoire à l'existence de coins Φ centrés en un élément général de N_α qui ne se relèvent pas à Y. Si on suppose qu'un tel coin Φ existe, en considérant la suite minimale d'éclatements de points fermés $\widetilde{W} \to \operatorname{Spec} K[[\xi, t]]$ qui "résolve Φ" (voir (3)), on déduit qu'il existe un noeud β_j de Γ, qui est non minimal et decroissant en $ch(\beta_j, \alpha)$ (déf. après corol. 2.2), et un k-morphisme $\widetilde{\Phi} : \mathbb{P}^1_K \to \mathbb{P}^1_k \cong E_{\beta_j}$. Si car $k = 0$, on obtient une contradiction à l'existence de Φ à partir du théorème de Hurwitz et de la forme trinomiale des équations d'Okuma [7] pour le revêtement abélien universel de (S, P_0).

Comme conséquence du th. 0.1 et de [8] on obtient que, si car $k = 0$, alors l'application de Nash $\mathcal{N}_X$ est bijective pour toute surface X sur k dont la normalisation n'a que des singularités rationnelles.

Une deuxième conséquence du th. 0.1, en appliquant [2,6] et [8], est : Soit car $k = 0$ et k non dénombrable. Si l'application de Nash $\mathcal{N}_{\overline{X}}$ est surjective pour toute surface normale sur k qui n'a que des singularités quasi-rationnelles (et non rationnelles), alors l'application de Nash $\mathcal{N}_X$ est surjective pour toute surface sur k.

Exemple 0.2. Soit car $k = 2$ et X la surface $x^3 + y^5 + z^2 = 0$ sur k. Pour $\lambda \in k \setminus \{0, 1\}$ le k-coin Φ_λ donné par $x(\xi, t) = t^4(\xi + \lambda t)^6$, $y(\xi, t) = t^2\ (\xi + \lambda t)^4$, $z(\xi, t) = t^5\ (\xi + \lambda t)^9\ (\xi + (1 + \lambda)t)$ est centré en un élément général de N_α, où ν_α est essentielle, et ne se relève pas à la désingularisation minimale de X. Cela donne un contre-exemple à la propriété de relèvement de k-coins par rapport à E_α, mais la propriété de relèvement de K-coins centrés en P_α est satisfaite. L'application de Nash $\mathcal{N}_X$ est bijective dans cet exemple.

1. Characteristic vectors of order functions

1.1. Let k be an algebraically closed field. Let (S, P_0) be a rational surface singularity over k, i.e. $S = \operatorname{Spec} R$ where R is a Noetherian normal complete two-dimensional local ring whose residue field is k and such that $R^1\pi'_*\mathcal{O}_{Y'} = 0$ for every desingularization $\pi' : Y' \to S$. Let $\pi : Y \to S$ be the minimal desingularization of (S, P_0), let $\{E_\alpha\}_{\alpha \in \Lambda}$ be the irreducible components of the exceptional locus of π, called *exceptional curves* of π, and $\{\nu_\alpha\}_{\alpha \in \Lambda}$ the divisorial valuations defined by these exceptional curves. Let $\mathbb{E}_Y := \bigoplus_{\alpha \in \Lambda} \mathbb{Z}\ E_\alpha$ and $\mathbb{E}^+_Y := \{D \in \mathbb{E}_Y / D \cdot E_\alpha \leqslant 0\ \forall \alpha \in \Lambda\}$. Since the intersection matrix $(E_\alpha \cdot E_{\alpha'})_{\alpha,\alpha'}$ is negative definite, for each $\alpha \in \Lambda$ there exists a unique $\mathbb{Q}$-divisor $\Delta_\alpha \in \mathbb{E}_Y \otimes \mathbb{Q}$ such that $\Delta_\alpha \cdot E_{\alpha'} = -\delta_{\alpha,\alpha'}$ for any $\alpha' \in \Lambda$. Let d_α be the smallest positive integer such that $d_\alpha \Delta_\alpha \in \mathbb{E}_Y$.

Let Γ be the dual graph of the exceptional curves of π. Let $\{\epsilon_i\}_{i=1}^m \subset \Lambda$ be the ends of Γ, and $\{\beta_j\}_{j=1}^s \subset \Lambda$ the nodes of Γ. We define the *extended dual graph* $\boldsymbol{\Gamma}$ *of* π to be the graph obtained from Γ by adding vertices i and edges joining i with ϵ_i for $1 \leqslant i \leqslant m$, and we denote by $\boldsymbol{\Lambda}$ the set of vertices of $\boldsymbol{\Gamma}$, i.e. $\boldsymbol{\Lambda} = \Lambda \cup \{1, \ldots, m\}$. Given $\alpha \in \Lambda$, let $\omega_\alpha := (-\Delta_\alpha \cdot \Delta_{\epsilon_1}, \ldots, -\Delta_\alpha \cdot \Delta_{\epsilon_m}) \in \mathbb{Q}^m_{\geqslant 0}$, and let $\operatorname{adj}^\Gamma(\alpha)$ (resp. $\operatorname{adj}^{\boldsymbol{\Gamma}}(\alpha)$) be the set of elements $\alpha' \in \Lambda$ (resp. $\gamma' \in \boldsymbol{\Lambda}$) which are adjacent to α in Γ (resp. in $\boldsymbol{\Gamma}$). For $1 \leqslant i \leqslant m$, let $\omega_i := (0, \ldots, 1, \ldots, 0)$ with the 1 in the i-th position. Then $E^2_\alpha \omega_\alpha + \sum_{\gamma \in \operatorname{adj}^{\boldsymbol{\Gamma}}(\alpha)} \omega_\gamma = 0$ for every $\alpha \in \Lambda$.

For $1 \leqslant i \leqslant m$, let $\mathcal{C}_i$ be a nonsingular irreducible curve in Y intersecting transversally the end exceptional curve E_{ϵ_i} in a point not belonging to any other exceptional curve. By Artin's property in [1, p. 133], there exists $x_i \in R$ such that $\operatorname{div}_Y(x_i) = d_{\epsilon_i}(\mathcal{C}_i + \Delta_{\epsilon_i})$. For $\alpha \in \Lambda$, let E^0_α be the open subset of E_α consisting of the (scheme theoretic) points of E_α which are neither in $E_{\alpha'}$ for $\alpha' \in \operatorname{adj}^\Gamma(\alpha)$, nor in $\bigcup_{i=1}^m \mathcal{C}_i$. For $1 \leqslant i \leqslant m$, set $\mathcal{C}^0_i := \mathcal{C}_i \setminus (\mathcal{C}_i \cap E_{\epsilon_i})$.

An *order function* ν on R is a function $\nu : R \to \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$ such that $\nu(\lambda) = 0$ for $\lambda \in k \setminus \{0\}$, $\nu(0) = +\infty$, $\nu(xy) = \nu(x) + \nu(y)$ and $\nu(x + y) \geqslant \min\{\nu(x), \nu(y)\}$ for $x, y \in R$. It defines a valuation on $R/\wp_\nu$ where $\wp_\nu = \{x \in R\ /\ \nu(x) = +\infty\}$; the center of ν is the center of this valuation. For any order function ν on R, we define the *characteristic vector of* ν *with respect to* $\mathcal{C}_1, \ldots, \mathcal{C}_m$ (for simplicity called characteristic vector of ν) to be $\omega_\nu := (\frac{\nu(x_1)}{d_{\epsilon_1}}, \ldots, \frac{\nu(x_m)}{d_{\epsilon_m}}) \in (\mathbb{Q}_{\geqslant 0} \cup \{+\infty\})^m$. For

$\alpha \in \Lambda$, the characteristic vector of ν_α is ω_α. For $1 \leqslant i \leqslant m$, let ν_i be discrete valuation of $K(R)$ whose center on Y is $\mathcal{C}_i$, its characteristic vector is ω_i.

Theorem 1.1. *Let ν be an order function on R such that $\omega_\nu \in (\mathbb{Q}_{\geqslant 0})^m \setminus \{\underline{0}\}$. Then, either there exists $\gamma_1 \in \boldsymbol{\Lambda}$ such that the center of ν on Y is contained in $E^0_{\gamma_1}$ (resp. in $\mathcal{C}^0_i$) if $\gamma_1 \in \Lambda$ (resp. if $\gamma_1 = i$), or there exist $\gamma_1, \gamma_2 \in \boldsymbol{\Lambda}$, $\gamma_2 \in \mathrm{adj}^{\boldsymbol{\Gamma}}(\gamma_1)$, such that the center of ν on Y is the point $E_{\gamma_1} \cap E_{\gamma_2}$ (resp. $\mathcal{C}_i \cap E_{\epsilon_i}$) if $\gamma_1, \gamma_2 \in \Lambda$ (resp. if $\{\gamma_1, \gamma_2\} = \{i, \epsilon_i\}$). In the first (resp. second) case, there exists a unique $n_{\gamma_1} \in \mathbb{N}$ (resp. unique $n_{\gamma_1}, n_{\gamma_2} \in \mathbb{N}$) such that $\omega_\nu = n_{\gamma_1}\, \omega_{\gamma_1}$ (resp. $\omega_\nu = n_{\gamma_1}\, \omega_{\gamma_1} + n_{\gamma_2}\, \omega_{\gamma_2}$).*

Conversely, if $\omega_\nu = n_{\gamma_1}\omega_{\gamma_1}$ where $\gamma_1 \in \boldsymbol{\Lambda}$ and $n_{\gamma_1} \in \mathbb{N}$ (resp. $\omega_\nu = n_{\gamma_1}\omega_{\gamma_1} + n_{\gamma_2}\omega_{\gamma_2}$ where $\gamma_1, \gamma_2 \in \boldsymbol{\Lambda}$, $\gamma_2 \in \mathrm{adj}^{\boldsymbol{\Gamma}}(\gamma_1)$ and $n_{\gamma_1}, n_{\gamma_2} \in \mathbb{N}$), then the center of ν on Y is contained in $E^0_{\gamma_1}$, or $\mathcal{C}^0_i$ if $\gamma_1 = i$, (resp. in $E_{\gamma_1} \cap E_{\gamma_2}$, or $\mathcal{C}_i \cap E_{\epsilon_i}$ if $i \in \{\gamma_1, \gamma_2\}$).

Idea of the proof. This is a result on values for divisorial valuations ν, it can be proved by induction on the number of point blowing ups $Y' \to Y$ such that the center of ν on Y' is a curve. The fan structure of the data $\{\omega_\gamma\}_{\gamma \in \boldsymbol{\Lambda}}$ is also used. □

2. Factors of a wedge

2.1. Let X be a surface over k (i.e. a reduced separated k-scheme of finite type which is equidimensional of dimension 2). For any field extension $k \subseteq K$, a *K-arc on X* is a k-morphism $\operatorname{Spec} K[\![t]\!] \to X$. There exists a k-scheme X_∞ called the *space of arcs* of X, whose K-rational points are the K-arcs on X, for any $K \supseteq k$. For $P \in X_\infty$, its image by the natural projection $j_0 : X_\infty \to X$ is called the *center of P*. We define $X^{\mathrm{Sing}}_\infty := j_0^{-1}(\operatorname{Sing} X)$. A *$K$-wedge on X* is a k-morphism $\Phi : \operatorname{Spec} K[\![\xi, t]\!] \to X$. It determines univocally a k-morphism $\varphi : \operatorname{Spec} K[\![\xi]\!] \to X_\infty$. The image in X_∞ of the closed point (resp. generic point) of $\operatorname{Spec} K[\![\xi]\!]$ by φ is called the *special arc* (resp. *generic arc*) of Φ. We say that Φ is centered at its special arc.

Let ν_α be an *essential divisor for X*, i.e. it is the divisorial valuation defined by an exceptional curve E_α for the minimal desingularization $\widetilde{X} \to X$ of X. Let N_α be the closure of the image by $\widetilde{X}_\infty \to X_\infty$ of the set of arcs on $\widetilde{X}$ whose center is on E_α. The set N_α is an irreducible subset of X^{Sing}_∞, let P_α be the generic point of N_α. We call P_α the *stable point* of X_∞ defined by ν_α (see [9, Definition 3.6]).

2.2. Let $P_0 \in X$ and suppose that the formal neighborhood of P_0 on X is a rational surface singularity (S, P_0). We identify the arcs on X centered at P_0 with the arcs $\operatorname{Spec} K[\![t]\!] \to S$ on (S, P_0), i.e. the elements of S_∞. For $P \in S_\infty$, defining an arc $h_P : \operatorname{Spec} \kappa(P)[\![t]\!] \to S$, let ν_P be the order function $\operatorname{ord}_t h^\sharp_P : R \to \mathbb{N} \cup \{\infty\}$. Given a K-wedge $\Phi : \operatorname{Spec} K[\![\xi, t]\!] \to S$, for each irreducible element p in $K[\![\xi, t]\!]$, let $\nu_{\Phi,p}$ be the order function $\operatorname{ord}_p \Phi^\sharp : R \to \mathbb{N} \cup \{\infty\}$. If $\omega_{\nu_{\Phi,p}} \in (\mathbb{Q}_{\geqslant 0})^m$, let $\omega_{\nu_{\Phi,p}} = \sum_{\gamma \in \boldsymbol{\Lambda}} n_\gamma(p)\, \omega_\gamma$ be the decomposition obtained applying Theorem 1.1 to $\nu_{\Phi,p}$. Here $n_\gamma(p) \in \mathbb{N} \cup \{0\}$ and $n_\gamma(p) > 0$ if and only if the center on Y of $\nu_{\Phi,p}$ is contained in E_γ (resp. in $\mathcal{C}_i$) if $\gamma \in \Lambda$ (resp. if $\gamma = i$), hence $n_\gamma(p) > 0$ for at most two elements γ_1, γ_2 which are adjacent in $\boldsymbol{\Gamma}$. If Φ is centered at $P \in S_\infty$ and $\omega_{\nu_P} \in (\mathbb{Q}_{\geqslant 0})^m$, hence $\omega_{\nu_{\Phi,p}} \in (\mathbb{Q}_{\geqslant 0})^m$ for every p, we define *the factors of Φ (with respect to $\mathcal{C}_1, \ldots, \mathcal{C}_m$)* to be $q_\gamma := \prod_p p^{n_\gamma(p)} \in K[\![\xi, t]\!]$ for $\gamma \in \Lambda$, where p runs over the set of irreducible elements of $K[\![\xi, t]\!]$ modulo product by a unit. There exist units o_i such that

$$x_i(\xi, t) := \Phi^\sharp(x_i) = o_i \prod_{\gamma \in \boldsymbol{\Lambda}} q_\gamma^{\nu_\gamma(x_i)} \in K[\![\xi, t]\!] \quad \text{for } 1 \leqslant i \leqslant m. \tag{1}$$

2.3. A point P of S_∞ is said to be a *general element of N_α* (resp. a general element of N_α with respect to $\mathcal{C}_1, \ldots, \mathcal{C}_m$) if it is the image by π_∞ of a point $Q \in Y_\infty$ centered at a point in $E_\alpha \setminus \bigcup_{\alpha' \neq \alpha} E_{\alpha'}$ (resp. in E^0_α) and whose induced arc h_Q intersects E_α transversally. For instance, P_α is a general element of N_α. If Φ is centered at a general element of N_α with respect to $\mathcal{C}_1, \ldots, \mathcal{C}_m$, then (1) implies

$$\omega_\alpha = \sum_{\gamma \in \boldsymbol{\Lambda}} \operatorname{ord}_t q_\gamma(0, t) \omega_\gamma. \tag{2}$$

For $\Delta \in \mathbb{E}_Y \otimes \mathbb{Q}$, let $\nu_{\alpha'}(\Delta) := \operatorname{coef}_{E_{\alpha'}} \Delta$, $\nu_i(\Delta) := -\Delta \cdot E_{\epsilon_i}$, for $\alpha' \in \Lambda$, $1 \leqslant i \leqslant m$.

Proposition 2.1. *If $\Phi : \operatorname{Spec} K[\![\xi, t]\!] \to S$ is a K-wedge centered at a general element of N_α with respect to $\mathcal{C}_1, \ldots, \mathcal{C}_m$, then Φ lifts to the minimal desingularization Y if and only if q_γ is a unit for every $\gamma \in \boldsymbol{\Lambda} \setminus \{\alpha\}$.*

Idea of the proof. This is a result on the graded algebra $gr_{\nu_\alpha} R$. It is a consequence of the following isomorphisms $\wp_{\alpha,n_\alpha}/\wp^+_{\alpha,n_\alpha} \cong \Gamma(Y, \mathcal{O}_{E_\alpha}(d_\alpha)) \cong k[T_0, T_1]_{d_\alpha}$, where $\wp_{\alpha,n_\alpha} = \Gamma(Y, \mathcal{O}_Y(-d_\alpha \Delta_\alpha))$, $n_\alpha = d_\alpha \nu_\alpha(\Delta_\alpha)$ and $\wp^+_{\alpha,n_\alpha} := \{f \in R / \nu_\alpha(f) > n_\alpha\}$. □

Corollary 2.2. *Suppose that there exists an end ϵ_i of Γ such that $\nu_\alpha(\Delta_{\epsilon_i}) < \nu_\gamma(\Delta_{\epsilon_i})$ for every $\gamma \in ch(\alpha, \epsilon_i) \setminus \{\alpha\}$. Then, every K-wedge Φ centered at a general element of N_α lifts to the minimal desingularization Y of X.*

A node β_j is said to be *nonminimal* if $E^2_{\beta_j} = -(m_j - 1)$ where $m_j := \sharp \operatorname{adj}^{\Gamma}(\beta_j)$. The singularity (S, P_0) is a minimal surface singularity iff Γ does not have nonminimal nodes. We say that β_j is *decreasing in* $ch(\beta_j, \alpha)$ if $\nu_{\alpha_{r-1}}(\Delta_\alpha) > \nu_{\alpha_r}(\Delta_\alpha)$ for $1 < r \leqslant n$, where $ch(\beta_j, \alpha) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, being $\alpha_1 = \beta_j$ and $\alpha_r \in \operatorname{adj}^{\Gamma}(\alpha_{r-1})$.

Let Φ be a K-wedge centered at a general element of N_α which does not lift to Y. There exists a finite sequence of blowing-ups of closed points $\widetilde{\rho} : \widetilde{W} \to \operatorname{Spec} K[\![\xi, t]\!]$ and a k-morphism $\widetilde{\Phi} : \widetilde{W} \to Y$ such that the following diagram is commutative

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\ \widetilde{\Phi}\ } & Y \\ \big\downarrow{\scriptstyle \widetilde{\rho}} & & \big\downarrow{\scriptstyle \pi} \\ W_0 = \operatorname{Spec} K[\![\xi, t]\!] & \xrightarrow{\ \Phi\ } & S. \end{array} \tag{3}$$

Let $\widetilde{W}$ be minimal with this property, we have $\widetilde{W} \neq W_0$ since Φ does not lift to Y. Applying the arguments in this section, and since Y is the minimal desingularization, it can be proved that there exists a nonminimal node β_j which is decreasing in $ch(\beta_j, \alpha)$ and such that E_{β_j} is the image by $\widetilde{\Phi}$ of an exceptional curve for $\widetilde{\rho}$.

3. The main result

Given a nonminimal node β_j and $\alpha_{j,l} \in \operatorname{adj}^{\Gamma}(\beta_j)$, $1 \leqslant l \leqslant m_j$, Theorem 4.2 in [4] implies that $\nu_{\beta_j}(\Delta_{\beta_j}) > \nu_{\alpha_{j,l}}(\Delta_{\beta_j})$. Let $\Lambda_{j,l}$ (resp., $\boldsymbol{\Lambda}_{j,l}$) be the set of vertices of the connected component of $\Gamma \setminus \{\beta_j\}$ (resp. $\boldsymbol{\Gamma} \setminus \{\beta_j\}$) which contains $\alpha_{j,l}$. Let $\Lambda^*_{j,l} = \Lambda_{j,l} \cup \{\beta_j\}$. From [4] it follows that there exists a divisor $D_{j,l}$ in $\mathbb{E}_{\Lambda_{j,l}} := \bigoplus_{\alpha' \in \Lambda_{j,l}} \mathbb{Z}\, E_{\alpha'}$ such that $\nu_{\alpha_{j,l}}(D_{j,l}) = 1$, $D_{j,l} \cdot E_{\alpha'} = 0$ for $\alpha' \in \Lambda_{j,l} \setminus \{\epsilon_1, \ldots, \epsilon_m\}$, and $D_{j,l} \cdot E_{\epsilon_i} \leqslant 0$ for $\epsilon_i \in \Lambda_{j,l}$. It also follows that there exists $D^*_{j,l} \in \mathbb{E}_{\Lambda^*_{j,l}} := \bigoplus_{\alpha' \in \Lambda^*_{j,l}} \mathbb{Z}\, E_{\alpha'}$ such that $\nu_{\beta_j}(D^*_{j,l}) = \nu_{\alpha_{j,l}}(D^*_{j,l}) = 1$, $D^*_{j,l} \cdot E_{\alpha'} = 0$ for $\alpha' \in \Lambda_{j,l} \setminus \{\epsilon_1, \ldots, \epsilon_m\}$, and $D^*_{j,l} \cdot E_{\epsilon_i} \leqslant 0$ for $\epsilon_i \in \Lambda_{j,l}$. If there are no nodes in $\Lambda_{j,l}$ then there exist unique $D_{j,l}$ and $D^*_{j,l}$ with the previous properties, and $(\nu_{\beta_j}(\Delta_{\beta_j}) - \nu_{\alpha_{j,l}}(\Delta_{\beta_j}))D_{j,l} = \nu_{\beta_j}(\Delta_{\beta_j})\, D^*_{j,l} - \Delta_{\beta_j}|_{\Lambda^*_{j,l}}$ where $\Delta_{\beta_j}|_{\Lambda^*_{j,l}}$ is the restriction of Δ_{β_j} to $\mathbb{E}_{\Lambda^*_{j,l}}$.

Theorem 3.1. *Let ν_α be an essential divisor for a rational surface singularity (S, P_0) over an algebraically closed field of characteristic 0. Let $\Phi : \operatorname{Spec} K[\![\xi, t]\!] \to S$ be a K-wedge centered at a general element of N_α. Then Φ lifts to the minimal desingularization Y of (S, P_0).*

Proof. Choose $\mathcal{C}_1, \ldots, \mathcal{C}_m$ not intersecting the strict transform of the special arc of Φ. Let $\{q_\gamma\}_{\gamma \in \Lambda}$ be the factors of Φ. After using Puiseux's theorem, we may suppose that $\operatorname{ord}_t p(0, t) = 1$ for every irreducible p dividing any of the q_γ's, and that K is algebraically closed. We argue by contradiction: suppose that Φ does not lift to Y. Then there exists a nonminimal node β_j which is decreasing in $ch(\beta_j, \alpha)$ and a point O' in the tree of points infinitely near O defining $\widetilde{\rho}$ in (3) such that the strict transform $C_{O'}$ in $\widetilde{W}$ of the exceptional curve of the blowing up of O' is sent by $\widetilde{\Phi}$ onto E_{β_j}; let O' be maximal with this property in the tree. The point O' is free by the condition $\operatorname{ord}_t p(0, t) = 1$. Let u, v be a regular system of parameters of the local ring at O' such that the exceptional locus of $W_{O'} := \operatorname{Spec} K[\![u, v]\!] \to W_0$ is contained in $u = 0$, and, for each $\gamma \in \boldsymbol{\Lambda}$, let $q'_\gamma \in K[\![u, v]\!]$ defining the strict transform of the curve $q_\gamma = 0$. Let $\Phi_{O'} : W_{O'} \to S$ be the K-wedge induced by Φ; its factors are $q_\gamma(\Phi_{O'}) = u^{n_\gamma(u)} q'_\gamma$, $\gamma \in \boldsymbol{\Lambda}$, where $\omega_{\Phi_{O'}, u} = \sum_\gamma n_\gamma(u)\omega_\gamma$ is the decomposition obtained applying Theorem 1.1 to $\nu_{\Phi_{O'}, u}$. Hence, there exists l_0, $1 \leqslant l_0 \leqslant m_j$, such that, for $l \neq l_0$, $n_\gamma(u) = 0$ for every $\gamma \in \boldsymbol{\Lambda}_{j,l}$.

From Okuma's equations for the universal abelian covering of (S, P_0) in [7] it follows that, for each three different $l_1, l_2, l_3 \in \{1, \ldots, m_j\}$, the following equality holds in $K[\![u, v]\!]$:

$$\lambda_1\, in\big(Q_{j,l_1}(\Phi_{O'})\big) + \lambda_2\, in\big(Q_{j,l_2}(\Phi_{O'})\big) + \lambda_3\, in\big(Q_{j,l_3}(\Phi_{O'})\big) = 0$$

where $\lambda_1, \lambda_2, \lambda_3 \in K$, $Q_{j,l}(\Phi_{O'}) = \prod_{\gamma \in \boldsymbol{\Lambda}_{j,l} \setminus \{\beta_j\}} q_\gamma(\Phi_{O'})^{\nu_\gamma(D_{j,l})}$ for $1 \leqslant l \leqslant m_j$ and *in* denotes the initial form in $K[\![u, v]\!]$. Besides, if $\varphi_{O'} : C_{O'} \to E_{\beta_j}$ is the restriction of $\widetilde{\Phi}$ and $d_{O'}$ is the degree of $\varphi^*_{O'}(\mathcal{O}_{E_{\beta_j}}(1))$, then $\operatorname{mult}_{O'} Q_{j,l}(\Phi_{O'}) = d_{O'}$ for $1 \leqslant l \leqslant m_j$. By [3, Lemma 7.1.2] (it can also be deduced from Hurwitz's theorem), we have $2d_{O'} - 2 \geqslant \sum_{1 \leqslant l \leqslant m_j}(d_{O'} - n_{j,l})$, where $n_{j,l}$ is the number of different linear factors of $in(Q_{j,l}(\Phi_{O'}))$. On the other hand, from (2), and applying that β_j is decreasing in $ch(\beta_j, \alpha)$, we conclude that $\nu_{\beta_j}(\Delta_{\beta_j}) \geqslant \sum_{\gamma \in \boldsymbol{\Lambda} \setminus \{\beta_j\}} \operatorname{mult}_{O'} q'_\gamma\ \nu_\gamma(\Delta_{\beta_j}) + d_{O'}$. Finally from the existence of the divisors $D^*_{j,l}$ and Okuma's equations at the nodes different to β_j we obtain $(\nu_{\beta_j}(\Delta_{\beta_j}) - \nu_{\alpha_{j,l}}(\Delta_{\beta_j})) \sum_{\gamma \in \boldsymbol{\Lambda}_{j,l} \setminus \{\beta_j\}} m_\gamma \nu_\gamma(D_{j,l}) = \nu_{\beta_j}(\Delta_{\beta_j}) \sum_{\gamma \in \boldsymbol{\Lambda}_{j,l} \setminus \{\beta_j\}} m_\gamma \nu_\gamma(D^*_{j,l}) - \sum_{\gamma \in \boldsymbol{\Lambda}_{j,l} \setminus \{\beta_j\}} m_\gamma \nu_\gamma(\Delta_{\beta_j})$ where $m_\gamma = \operatorname{mult}_{O'} q_\gamma(\Phi_{O'}) = \operatorname{mult}_{O'} q'_\gamma + n_\gamma(u)$. Set $e_{l_0} = 1$ if there exists $\gamma \in \boldsymbol{\Lambda}_{j,l_0}$ such that $n_\gamma(u) > 0$, otherwise $e_{l_0} = 0$. Then we conclude that

$$\nu_{\beta_j}(\Delta_{\beta_j}) \geqslant \nu_{\beta_j}(\Delta_{\beta_j})\big((m_j - 3)d_{O'} + 2 - e_{l_0}\big) + e_{l_0}\big(\nu_{\beta_j}(\Delta_{\beta_j}) - \nu_{\alpha_{j,l_0}}(\Delta_{\beta_j})\big)$$

and, since $m_j \geqslant 3$ and $\nu_{\beta_j}(\Delta_{\beta_j}) > \nu_{\alpha_{j,l_0}}(\Delta_{\beta_j})$, a contradiction follows.

From Theorem 3.1 and applying [8] Theorem 5.1, [9] Corollaries 5.12, 5.15 and [2] satzs 1.7, 2.8, [6] Proposition 4.2, we conclude Corollaries 3.2 and 3.3. Recall that a normal surface singularity is a *quasirational singularity* if it has a desingularization whose exceptional curves are rational curves. Given a domain A, we denote by $\overline{A}$ the integral closure of A in its quotient field. □

Corollary 3.2. *Suppose that* $\operatorname{char} k = 0$. *Let X be a surface over k whose normalization has only rational surface singularities. Then, for every essential divisor ν_α over X, every K-wedge centered at P_α lifts to the minimal desingularization of X. Thus, the Nash map $\mathcal{N}_X$ is bijective, the rings $A_\alpha := \mathcal{O}_{X_\infty, P_\alpha}$ have dimension* 1, *and $(\widehat{A_\alpha})_{\mathrm{red}}$ and $\overline{(A_\alpha)_{\mathrm{red}}}$ are regular local rings.*

Corollary 3.3. *Suppose that k is uncountable and* $\operatorname{char} k = 0$. *If the Nash map $\mathcal{N}_{\overline{X}}$ is surjective for every irreducible normal surface $\overline{X}$ over k having quasirational singularities which are not rational singularities, then the Nash map $\mathcal{N}_X$ is surjective for every surface X over k.*

Example 3.4. Suppose that $\operatorname{car} k = 2$ and let X be the surface $x^3 + y^5 + z^2 = 0$ over k, which has an $\mathbb{E}_8$ singularity at the origin. Let E_α be the exceptional curve for the minimal desingularization $\widetilde{X}$ corresponding to the unique node of its dual graph. For $\lambda \in k \setminus \{0, 1\}$ the k-wedge $\Phi_\lambda : \operatorname{Spec} k[\![\xi, t]\!] \to (S, P_0)$ given by $x(\xi, t) := \Phi_\lambda^\sharp(x) = t^4(\xi + \lambda t)^6$, $y(\xi, t) := \Phi_\lambda^\sharp(y) = t^2(\xi + \lambda t)^4$, $z(\xi, t) := \Phi_\lambda^\sharp(z) = t^5(\xi + \lambda t)^9(\xi + (1 + \lambda)t)$ does not lift to $\widetilde{X}$. This shows a counterexample to the property of lifting k-wedges with respect to E_α to the minimal desingularization [6, Definition 2.10]. The Nash map $\mathcal{N}_X$ is bijective in this example.

References

[1] M. Artin, On isolated rational singularities of surfaces, Amer. J. Math. 88 (1966) 129–136.
[2] E. Brieskorn, Rationale Singularitäten komplexer Flächen, Inv. Math. 4 (1968) 336–358.
[3] E. Casas-Alvero, Singularities of Plane Curves, London Math. Soc. Lecture Note, vol. 276, Cambridge University Press, 2000.
[4] H. Laufer, On rational singularities, Amer. J. Math. 94 (1972) 597–608.
[5] M. Lejeune-Jalabert, A. Reguera, Arcs and wedges on sandwiched surface singularities, Amer. J. Math. 121 (1999) 1191–1213.
[6] M. Lejeune-Jalabert, A. Reguera, Exceptional divisors which are nor uniruled belong to the image of the Nash map, arXiv:0811.2421v1 (2008), J. Inst. Math. Jussieu, in press.
[7] T. Okuma, Universal Abelian covers of certain surface singularities, Math. Ann. 334 (2006) 753–773.
[8] A.J. Reguera, A curve selection lemma in spaces of arcs and the image of the Nash map, Compositio Math. 142 (2006) 119–130.
[9] A.J. Reguera, Towards the singular locus of the space of arcs, Amer. J. Math. 131 (2) (2009) 313–350.