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Partial Differential Equations/Functional Analysis

Fractional powers approach of operators for the solvability of some elliptic PDE's with variable operators coefficients ☆

Approche utilisant les puissances fractionnaires d'opérateurs dans la résolution de quelques EDP elliptiques à coefficients opérateurs variables

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ABSTRACT

This Note is devoted to the study of a complete second order differential equation of elliptic type with variable operators coefficients and Dirichlet inhomogeneous boundary conditions. We give necessary and sufficient conditions on the data for the existence and uniqueness of the strict solution by using some natural differentiability assumptions on the resolvent operators of the square roots characterizing the ellipticity. The techniques used here are essentially based on the semigroups theory and the fractional powers of linear operators.

RÉSUMÉ

Cette Note est consacrée à l'étude d'une équation différentielle complète du second ordre de type elliptique et à coefficients opérateurs variables avec des conditions aux limites de Dirichlet non homogènes. On donne des conditions nécessaires et suffisantes sur les données pour l'existence et l'unicité de la solution stricte en utilisant des hypothèses naturelles sur la différentiabilité des résolvantes des racines carrées des opérateurs caractérisant l'ellipticité. Les techniques utilisées ici sont basées essentiellement sur la théorie des semi-groupes et les puissances fractionnaires d'opérateurs linéaires.

Let us consider some partial differential equations (of elliptic or quasi-elliptic type) such as the two following models:

$$\frac{\partial^2 u}{\partial x^2}(x,y)+b(x,y)\frac{\partial u}{\partial x}(x,y)+a(x,y)\frac{\partial^2 u}{\partial y^2}(x,y)-\lambda u(x,y)=f(x,y),\quad x\in\,]0,1[,\ y\in\,]c,d[,$$

(as for the Laplace operator) or

$$\frac{\partial^2 u}{\partial x^2}(x,y)+b(x,y)\frac{\partial u}{\partial x}(x,y)-a(x,y)\frac{\partial^4 u}{\partial y^4}(x,y)-\lambda u(x,y)=f(x,y),\quad x\in\,]0,1[,\ y\in\,]c,d[,$$

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(as for the biharmonic operator), a being positive, a and b having Hölderian regularity with respect to x and some other with respect to y. Under various boundary Dirichlet–Neumann conditions with respect to the variable y (depending on x) and with data $u(0, y) = \varphi(y)$, $u(1, y) = \psi(y)$, this class of PDE's can be written, in some complex Banach space X, as the following complete second order differential equation with variable operator coefficients

$$u''(x) + B(x)u'(x) + A(x)u(x) - \lambda u(x) = f(x), \quad x \in (0,1), \tag{1}$$

$$u(0) = \varphi, \qquad u(1) = \psi. \tag{2}$$

In this problem λ is a positive real number, $f \in C^\theta([0,1]; X)$, $0 < \theta < 1$, φ and ψ are some given elements in X, $(B(x))_{x\in([0,1])}$ is a family of bounded linear operators and $(A(x))_{x\in([0,1])}$ is a family of closed linear operators in X, of domains $D(A(x))$ not necessarily dense in X. Put $Q(x) = A(x) - \lambda I$, $\lambda > 0$. Recently, Eq. (1) was treated by many authors: For the constant case $B(x) = B$ and $A(x) = A$, see [4].

In the variable case, for $x \in (0, \delta)$, δ a small real positive number, Eq. (1) with boundary conditions $u(0) = \varphi, u'(\delta) = \psi$ was treated in [5] by using the interpolation spaces, Dunford's functional calculus and some techniques given in [7]. In this Note, we give an alternative approach: for each $x \in [0,1]$, from assumption (3), the square roots $\sqrt{-Q(x)}$ are well defined and $K(x) = -\sqrt{-Q(x)}$ generates an analytic semigroup $(e^{yK(x)})_{y>0}$, not necessarily strongly continuous in 0 (see [2]). Therefore, we use this last important property, which was not considered to study Eq. (1), in order to analyze and improve the study in [5]. This will be done under some natural differentiability assumptions on the resolvent operators of $\sqrt{-Q(x)}$ extending the study by [1] in the parabolic case. We also use the Dunford's functional calculus. We obtain necessary and sufficient conditions for the existence and uniqueness of the strict solution, by using the square roots $-\sqrt{-Q(x)}$, while in [5] the authors give, only, a sufficient condition for the existence and uniqueness of the strict solution with the help of operators $Q(x)$. Observe that this equation (when $B(x) \equiv 0$) was treated in [3] by the sums of linear operators. In this work, we present some basic results concerning the properties of the exponential function $x \mapsto e^{-x\sqrt{-Q(x)}}$ and its derivatives which lead to prove our main result (Theorem 1) related to the existence and uniqueness of the strict solution of problem (1)–(2). Recall that a strict solution is a function u such that $u \in C^2([0,1]; X)$, $u(x) \in D(Q(x))$ for each $x \in [0,1]$, $x \mapsto Q(x)u(x) \in C([0,1]; X)$ and u satisfies problem (1)–(2). In all this work, we will use the following hypotheses

$$\exists C > 0,\ \forall z \geqslant 0,\ \forall x \in [0,1],\ \exists \big(Q(x) - zI\big)^{-1} \in \mathcal{L}(X) \quad \text{and} \quad \big\|\big(Q(x) - zI\big)^{-1}\big\|_{\mathcal{L}(X)} \leqslant \frac{C}{1+z}, \tag{3}$$

$$\exists C > 0\text{: } \forall x \in [0,1], \quad \big\|B(x)\big\|_{\mathcal{L}(X)} \leqslant C. \tag{4}$$

Here, the term $B(x)u'(x)$ is considered in Eq. (1) as a "perturbation". Due to (3), there exists a sector (for some small $\theta_1 > 0$ and $r_1 > 0$):

$$\Pi_{\theta_1+\frac{\pi}{2},r_1} = \left\{z \in \mathbb{C}^*\text{: } \big|\arg(z)\big| \leqslant \theta_1 + \frac{\pi}{2}\right\} \cup \big\{z \in \mathbb{C}\text{: } |z| = r_1\big\},$$

such that $\Pi_{\theta_1+\frac{\pi}{2},r_1} \subset \rho(-(-Q(x))^{\frac{1}{2}}) = \rho(K(x))$. Further to Assumptions (3) and (4), we will assume that: for all $z \in \Pi_{\theta_1+\frac{\pi}{2},r_1}$, the mapping $x \mapsto (K(x) - zI)^{-1}$, defined on $[0,1]$, is in $C^2([0,1], \mathcal{L}(X))$ and there exist $C > 0$, $\nu \in]1/2, 1]$ and $\eta \in]0,1[$ such that $\forall z \in \Pi_{\theta_1+\frac{\pi}{2},r_1}$, $\forall x, s \in [0,1]$,

$$\left\|\frac{\partial}{\partial x}\big(K(x) - zI\big)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{C}{|z|^\nu}, \tag{5}$$

$$\left\|\frac{\partial}{\partial x}\big(K(x) - zI\big)^{-1} - \frac{\partial}{\partial s}\big(K(s) - zI\big)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{C|x-s|^\eta}{|z|^\nu}, \quad \text{with } \eta + \nu - 1 > 0, \tag{6}$$

$$\left\|\frac{\partial^2}{\partial x^2}\big(K(x) - zI\big)^{-1}\right\|_{\mathcal{L}(X)} \leqslant C|z|^{1-\nu}, \qquad \left\|\frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(K(x)\big)^{-1} - \frac{\mathrm{d}^2}{\mathrm{d}s^2}\big(K(s)\big)^{-1}\right\|_{\mathcal{L}(X)} \leqslant C|x-s|^\eta, \tag{7}$$

$$B(0)(X) \subset \overline{D\big(K(0)\big)} = \overline{D\big(Q(0)\big)}, \qquad B(1)(X) \subset \overline{D\big(K(1)\big)} = \overline{D\big(Q(1)\big)}, \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(K(x)\big)^{-1}_{|x=0}\big(D\big(K(0)\big)\big) \subset \overline{D\big(K(0)\big)}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\big(K(x)\big)^{-1}_{|x=1}\big(D\big(K(1)\big)\big) \subset \overline{D\big(K(1)\big)}. \tag{9}$$

Now, we are interesting to the construction of the solution. As in [6], when $B(x) = 0$ and $Q(x) = Q$ is a constant operator satisfying (3), we will seek a solution u of problem (1)–(2) in the following form

$$u(x) = e^{xK(x)}\xi_0^*(x) + e^{(1-x)K(x)}\xi_1^*(x) + \frac{1}{2}\int_0^x e^{(x-s)K(x)}(K(x))^{-1} f^*(s)\,\mathrm{d}s + \frac{1}{2}\int_x^1 e^{(s-x)K(x)}\big(K(x)\big)^{-1} f^*(s)\,\mathrm{d}s, \tag{10}$$

where f^* is an unknown function in $C^\beta([0,1];X)$ (β is to be determined in $]0,1[$). The functions ξ_0^* and ξ_1^* are defined by (for the definition of $(I-Z(x))^{-1}$ with $Z(x)=e^{2K(x)}$, see [8], p. 60)

$$\xi_0^*(x) = \big(I-Z(x)\big)^{-1}\big(\varphi - e^{K(x)}\psi\big) - \frac{(I-Z(x))^{-1}}{2}\int_0^1 e^{sK(x)}\big(K(x)\big)^{-1} f^*(s)\,\mathrm{d}s$$
$$+ \frac{(I-Z(x))^{-1}}{2}\int_0^1 e^{(2-s)K(x)}\big(K(x)\big)^{-1} f^*(s)\,\mathrm{d}s,$$
$$\xi_1^*(x) = \big(I-Z(x)\big)^{-1}\big(\psi - e^{K(x)}\varphi\big) - \frac{(I-Z(x))^{-1}}{2}\int_0^1 e^{(1-s)K(x)}\big(K(x)\big)^{-1} f^*(s)\,\mathrm{d}s$$
$$+ \frac{(I-Z(x))^{-1}}{2}\int_0^1 e^{(1+s)K(x)}\big(K(x)\big)^{-1} f^*(s)\,\mathrm{d}s.$$

Proposition 1. *Let $\varphi \in D((K(0))^2)$ and assume (3)–(9). Then,*

1. *The function $x \mapsto e^{xK(x)}\varphi$ belongs to $C([0,1];X)$ if and only if $\varphi \in \overline{D(K(0))}$ and $\lim_{x\to 0} e^{xK(x)}\varphi = \varphi$.*
2. *The function $x \mapsto \frac{\mathrm{d}}{\mathrm{d}x}(e^{xK(x)})\varphi$ belongs to $C^{\min(\eta,\nu)}([0,1];X)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(e^{xK(x)})\varphi \to K(0)\varphi$, as $x \to 0$.*
3. *The function $x \mapsto \frac{\mathrm{d}^2}{\mathrm{d}x^2}(e^{xK(x)})\varphi$ belongs to $C^{\min(\eta,\nu)}([0,1];X)$. Moreover*

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(e^{xK(x)}\big)\varphi \to \big(K(0)\big)^2\varphi - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(K(x)\big)^{-1}_{|x=0}K(0)\varphi, \quad \text{as } x \to 0,$$

if and only if

$$\big(K(0)\big)^2\varphi - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(K(x)\big)^{-1}_{|x=0}K(0)\varphi \in \overline{D\big(K(0)\big)} = \overline{D\big(Q(0)\big)}.$$

4. *If $\psi \in D((K(1))^2)$ and $f \in C^\theta([0,1];X)$, $0<\theta<1$, then $f^* \in C^\beta([0,1];X)$, $\beta = \min(\theta, \eta+\nu-1)$ and*

$$f^*(0) = f(0) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(K(x)\big)^{-1}_{|x=0}K(0)\varphi + \Phi_0^*(\varphi) + r_0\big(f^*,\psi\big),$$
$$f^*(1) = f(1) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(K(x)\big)^{-1}_{|x=1}K(1)\psi + \Phi_1^*(\psi) + r_1\big(f^*,\varphi\big),$$

where $\Phi_0^*(\varphi), r_0(f^*,\psi) \in \overline{D(K(0))} = \overline{D(Q(0))}$ and $\Phi_1^*(\psi), r_1(f^*,\varphi) \in \overline{D(K(1))} = \overline{D(Q(1))}$.

Observe that the study of the regularity of u'' is based on the behavior of operators $e^{xK(x)}\varphi$, $e^{(1-x)K(x)}\psi$ and their derivatives. It is rather complicated to study it near 0 and 1. This leads us to adopt another strategy by introducing new operators written in terms of semigroups and treating them very carefully in 0 and 1. We must have to establish the convergence of all integrals obtained here using some techniques as in [1,5] and [9]. Our main result on the existence and uniqueness of the strict solution is the following:

Theorem 1. *Let $\varphi \in D((K(0))^2)$, $\psi \in D((K(1))^2)$ and $f \in C^\theta([0,1];X)$, $0<\theta<1$. Then, under Hypotheses (3)–(9), there exists $\lambda^* > 0$ such that for all $\lambda \geqslant \lambda^*$, the function u given in the representation (10) is the unique strict solution of problem (1)–(2) if and only if*

$$\begin{cases} \big(K(0)\big)^2\varphi + f(0) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(K(x)\big)^{-1}_{|x=0}K(0)\varphi \in \overline{D\big(K(0)\big)} = \overline{D\big(Q(0)\big)}, \\ \big(K(1)\big)^2\psi + f(1) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(K(x)\big)^{-1}_{|x=1}K(1)\psi \in \overline{D\big(K(1)\big)} = \overline{D\big(Q(1)\big)}. \end{cases} \tag{11}$$

Sketch of the proof of Theorem 1. It is enough to consider the case $\psi = 0$ and prove that $x \mapsto Q(x)u(x) \in C([0,1];X)$ if and only if

$$\big(K(0)\big)^2\varphi + f(0) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(K(x)\big)^{-1}_{|x=0}K(0)\varphi \in \overline{D\big(K(0)\big)}.$$

Regarding, for instance, the following singular terms in 0 and occurring in the expression of $Q(x)u(x)$ one has:

$$\begin{aligned}
&\big(K(x)\big)^2 e^{xK(x)}\varphi + e^{xK(x)} f^*(0)\\
&\quad = \big[\big(K(x)\big)^2 - \big(K(0)\big)^2\big] e^{xK(x)}\varphi + \big(K(0)\big)^2\big[e^{xK(x)} - e^{xK(0)}\big]\varphi\\
&\qquad + \big[e^{xK(x)} - e^{xK(0)}\big] f^*(0) + \big(K(0)\big)^2 e^{xK(0)}\varphi + e^{xK(0)} f^*(0)\\
&\quad = (a) + (b) + (c) + (d) + (e).
\end{aligned}$$

By using Proposition 1, one obtains (a), (b) tend to 0, as x tends to 0 and $(c) \in C([0,1]; X)$.
In addition

$$\begin{aligned}
(d) + (e) &= e^{xK(0)}\left[\big(K(0)\big)^2\varphi + f(0) + \frac{d^2}{dx^2}\big(K(x)\big)^{-1}_{|x=0} K(0)\varphi\right] + e^{xK(0)}\Phi_0^*(\varphi) + e^{xK(0)} r_0\big(f^*, \psi\big)\\
&= (\alpha) + (\beta) + (\gamma).
\end{aligned}$$

Proposition 1 again leads to $(\beta), (\gamma) \in C([0,1]; X)$ and $(\alpha) \in C([0,1]; X)$ if and only if

$$\big(K(0)\big)^2\varphi + f(0) + \frac{d^2}{dx^2}\big(K(x)\big)^{-1}_{|x=0} K(0)\varphi \in \overline{D\big(K(0)\big)} = \overline{D\big(Q(0)\big)}.$$

Similarly, one gets the second compatibility condition in (11). Thus, one obtains new results using operators $\sqrt{-Q(x)}$. These results improve those related to Eq. (1) and proved in [5] by another method. □

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