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A new weighted Bellman function

Une nouvelle fonction pondérée de Bellman

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A R T I C L E I N F O

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A B S T R A C T

We give sharp in p and w estimates of operator norms of Riesz transforms in the $L^p(w\,dx)$ spaces, when the A_p characteristic of the weight is close to 1 (flat case). This is done by proving the existence of a certain Bellman function.

R É S U M É

Nous construisons une nouvelle fonction de Bellman qui nous permet de donner des estimations précises de la norme des transformées de Riesz dans les espaces pondérés $L^p(w\,dx)$, quand la caractéristique du poids est proche de 1.

Version française abrégée

Les estimations précises des normes des opérateurs de type intégrales singulières sont connues seulement dans très peu de cas. Le cas ouvert le plus célèbre est celui de la transformation de Ahlfors–Beurling $T = R_1^2 - R_2^2 + 2iR_1R_2$, où l'estimation précise serait d'un grand intérêt. Dans cet article nous construisons une nouvelle fonction de Bellman basée sur les résultats de [6]. Cette fonction peut être utilisée (voir section 2) pour estimer les normes de transformées des Riesz de deuxième ordre et pour donner une meilleure compréhension de l'estimation de Littlewood–Paley qui est apparue dans [3].

Par exemple, en utilisant notre théorème principal et les techniques de [3,5,7], nous pouvons montrer que pour tout $f, g \in C_c^\infty(\mathbf{R}^2)$ l'inégalité suivante est vraie ($p^* := \max, p, \frac{p}{p-1}$) :

$$2\int_0^{+\infty}\int_{\mathbf{R}^2}\left(\left|\frac{\partial f^h}{\partial x_1}\right| + \left|\frac{\partial f^h}{\partial x_2}\right|\right)^{\frac{1}{2}}\left(\left|\frac{\partial g^h}{\partial x_1}\right| + \left|\frac{\partial g^h}{\partial x_2}\right|\right)^{\frac{1}{2}} \mathrm{d}y\,\mathrm{d}t \leqslant (p^*-1)\left(1 + c(p)\sqrt{[w]_{A_p} - 1}\right)\|f\|_{L^p(w)}\|g\|_{L^{p'}(w^{1-p'})},$$

pour chaque poids w sur $\mathbf{R}^2$ de classe A_p tel que $[w]_{A_p} \leqslant 1 + \delta$, $\delta \approx 0$. Les fonctions du terme de gauche sont les extensions de chaleur de f, g, respectivement. Il faut comparer ce resultat avec celui de [7,3] :

$$2\int_0^{+\infty}\int_{\mathbf{R}^2}\left(\left|\frac{\partial f}{\partial x_1}\right| + \left|\frac{\partial f}{\partial x_2}\right|\right)^{\frac{1}{2}}\left(\left|\frac{\partial g}{\partial x_1}\right| + \left|\frac{\partial g}{\partial x_2}\right|\right)^{\frac{1}{2}} \mathrm{d}y\,\mathrm{d}t \leqslant C(p)[w]_{A_p}^{\max(1, \frac{1}{p-1})}\|f\|_{L^p(w)}\|g\|_{L^{p'}(w^{1-p'})},$$

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pour chaque poids $w \in A_p$. La différence est dans l'expression de la constante $C(p)$. On voit que pour $p = 2$ et $[w]_{A_p} - 1 = \delta \approx 0$ par exemple, la premier estimation donne l'ordre $1 + C\sqrt{\delta}$ (et c'est précis), alors que la seconde estimation donne $C(1+\delta)$ pour un certain C.

1. Introduction

In this Note we construct a new Bellman function based on results from [6]. It can be used (see Section 2) to estimate the norms of second order Riesz transforms and to give a better understanding of some Littlewood–Paley estimates that first appeared in [3]. For instance using our main theorem and techniques from [3,5,7], we can prove that for any $f, g \in C_c^\infty(\mathbf{R}^2)$ the following inequality is true:

$$2\int_0^{+\infty}\int_{\mathbf{R}^2}\left(\left|\frac{\partial f^h}{\partial x_1}\right| + \left|\frac{\partial f^h}{\partial x_2}\right|\right)^{\frac{1}{2}}\left(\left|\frac{\partial g^h}{\partial x_1}\right| + \left|\frac{\partial g^h}{\partial x_2}\right|\right)^{\frac{1}{2}} \mathrm{d}y\,\mathrm{d}t \leqslant \bigl(p^*-1\bigr)(1+c\sqrt{\delta})\|f\|_{L^p(w)}\|g\|_{L^{p'}(w^{1-p'})}$$

for any A_p weight w on $\mathbf{R}^2$, $[w]_{A_p} = \sup_Q(\frac{1}{|Q|}\int_Q w)(\frac{1}{|Q|}\int_Q w^{\frac{-1}{p-1}})^{p-1} \leqslant 1+\delta < 2$. The functions on the left-hand side are the heat extensions of f, g respectively, and c is a constant that depends on p.

2. Main theorem and applications

In [7], the following theorem was proved:

Theorem 2.1. *For any $Q > 1$, $p > 2$ define the domain $D_Q^p = \{0 < (X, Y, x, y, r, s) \in \mathbf{R}^6$: $x^p < Xs^{p-1}$, $y^{p'} < Yr^{p'-1}$, $1 < rs^{p-1} < Q\}$. Let K be any compact subset of D_Q^p. Then there exists a function $B = B_{Q,K}^{(p)}(X, Y, x, y, r, s)$ infinitely differentiable in a small neighborhood of K, such that*

(1) $0 \leqslant B \leqslant C(p)QX^{1/p}Y^{1/p'}$,
(2) $-d^2B \geqslant 2|\mathrm{d}x||\mathrm{d}y|$.

Unfortunately, we do not have any nice expression for the constant $C(p)$ that appears in (1) since in order to prove this in [7], the authors used extrapolation, and it is not easy to keep track of the constants since in the estimates the Hardy–Littlewood maximal function plays a fundamental role. Our main result is really similar to the previous one but it gives a really nice expression for $C(p)$ and behaves also nice as the constant Q goes to 1.

Theorem 2.2. *For any $1 < Q < 2$, $1 < p < +\infty$ define the domain $D_Q^p = \{0 < (X, Y, x, y, r, s) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}$: $|x|^p < Xs^{p-1}$, $|y|^{p'} < Yr^{p'-1}$, $1 < rs^{p-1} < Q\}$. Let K be any compact subset of D_Q^p. Then there exists a function $B = B_{Q,K}^{(p)}(X, Y, x, y, r, s)$ infinitely differentiable in a small neighborhood of K, and at the same time for any $\epsilon > 0$, $B_{Q,K}$ can be chosen in such a way that*

(1) $0 \leqslant B \leqslant (p^*-1)(1+\epsilon)(1+c\sqrt{\delta})X^{1/p}Y^{1/p'}$,
(2) $-d^2B \geqslant 2|\mathrm{d}x||\mathrm{d}y|$, *where $Q = 1+\delta$ and c is a constant that depends on p and the dimension d.*

Proof. In [6], it was shown that for an A_p weight w, on $\mathbf{R}$, of characteristic $[w]_{A_p} < 1+\delta < 2$,

$$\|T_r\|_{L^p(w)\to L^p(w)} \leqslant \|T_r\|_{L^p\to L^p}(1+c\sqrt{\delta}),$$

where c is a constant that depends on p. It is really easy to see that the interpolation in [8] works for the vectorized martingale transform. This means that using the techniques from [6], the above inequality is also true for the vectorized martingale transform (acting on functions with values in a separable Hilbert space). But a famous result by Burkholder (see [1], and an extension of it in [3]), states that $\|T_r\|_{L^p\to L^p} = p^*-1$. Therefore,

$$\|T_r\|_{L^p(w)\to L^p(w)} \leqslant \bigl(p^*-1\bigr)(1+c\sqrt{\delta}).$$

Now, by using duality we arrive to the point (we denote by $|\,.\,|$ the norm in our Hilbert space):

$$\frac{1}{4|J|}\sum_{I\in\mathcal{D}(J)}\bigl|\langle f\rangle_{I_+} - \langle f\rangle_{I_-}\bigr|\bigl|\langle g\rangle_{I_+} - \langle g\rangle_{I_-}\bigr||I| \leqslant \bigl(p^*-1\bigr)(1+c\sqrt{\delta})\bigl\langle|f|^p w\bigr\rangle_J^{1/p}\bigl\langle|g|^{p'}w^{1-p'}\bigr\rangle_J^{1/p'},$$

for any $J \in \mathcal{D}$, any vector functions $f \in L^p(w)$ and $g \in L^{p'}(w^{1-p'})$. The definition of the Bellman function is the following:

$$B(X,Y,x,y,r,s)=\sup\left\{\frac{1}{4|J|}\sum_{I\in\mathcal{D}(J)}\left|\langle f\rangle_{I_+}-\langle f\rangle_{I_-}\right|\left|\langle g\rangle_{I_+}-\langle g\rangle_{I_-}\right||I|:\right.$$
$$\left.\langle f\rangle_J=x,\ \langle g\rangle_J=y,\ \langle w\rangle_J=r,\ \left\langle w^{1-p'}\right\rangle_J=s,\ \left\langle |f|^p w\right\rangle_J=X,\ \left\langle |g|^{p'}w^{1-p'}\right\rangle_J=Y\right\}.$$

Obviously, this function satisfies inequality (1) in the statement of our theorem. We claim that for all 6-tuples $a^+=(X^+,Y^+,x^+,y^+,r^+,s^+)$, $a^-=(X^-,Y^-,x^-,y^-,r^-,s^-)\in D^p_Q$, such that $\frac{a^++a^-}{2}\in D^p_Q$, the following inequality is true (for the proof, follow the corresponding one in [7]):

$$B\left(\frac{a^++a^-}{2}\right)-\frac{B(a^+)+B(a^-)}{2}\geqslant\frac{1}{4}\left|x^+-x^-\right|\left|y^+-y^-\right|.$$

Now we need to mollify this function B, in order to take the smooth version of it. This can be done in exactly the same way as in [5]. The concavity inequality remains the same after the mollification and the size condition can become $1+C_K\epsilon$ times worse, where C_K is just a constant that depends on the compact set K. □

For a nice application of this theorem, we can formulate the following result:

Theorem 2.3. *Let $1<p<+\infty$, and any scalar A_p weight w on $\mathbf{R}^d$ of $[w]_{A_p}<1+\delta<2$,*

$$\|\mathcal{R}\|_{L^p(\mathbf{R}^d,\mathbf{R}^d,w\,\mathrm{d}x)\to L^p(\mathbf{R}^d,\mathbf{R}^d,w\,\mathrm{d}x)}\leqslant\left(p^*-1\right)(1+c\sqrt{\delta}),$$

where $\mathcal{R}=(R_iR_j)^d_{i,j=1}$ is a matrix with each entry a product of two Riesz transforms.

Observe that if we let δ go to 0, which means w becomes a constant weight, $\|\mathcal{R}\|_{L^p(\mathbf{R}^d,\mathbf{R}^d)\to L^p(\mathbf{R}^d,\mathbf{R}^d)}\leqslant(p^*-1)$. The proof of this theorem follows the standard techniques appearing in [3,5,7], in which the existence of the Bellman function implies a Littlewood–Paley type estimate and it, in its turn, implies the desired estimate.

Proof. Using the Bellman function of Theorem 2.2, we can show that for any A_p weight, w, on $\mathbf{R}^d$ of $[w]_{A_p}\leqslant 1+\delta<2$, and any *scalar* functions $\phi,\psi\in C_c^\infty(\mathbf{R}^d)$ the following is true:

$$2\int_{\mathbf{R}^{d+1}_+}\left(\sum_{i=1}^d\left|\frac{\partial\phi^h(x,t)}{\partial x_i}\right|^2\right)^{\frac12}\left(\sum_{i=1}^d\left|\frac{\partial\psi^h(x,t)}{\partial x_i}\right|^2\right)^{\frac12}\mathrm{d}x\,\mathrm{d}t\leqslant\left(p^*-1\right)(1+c\sqrt{\delta})\|\phi\|_{L^p(w)}\|\psi\|_{L^{p'}(w^{1-p'})},$$

where the functions appearing on the left-hand side are the heat extensions to $\mathbf{R}^{d+1}_+$, of ϕ,ψ. For example, for ϕ say,

$$\phi^h(x,t)=c_d\int_{\mathbf{R}^d}\phi(y)\exp\left(-\frac{|x-y|^2}{4t}\right)\mathrm{d}y.$$

In addition, expressing the norm of $\mathcal{R}$ by duality we obtain (here $\Phi=(\phi_j)^d_{j=1}$, $\Psi=(\psi_i)^d_{i=1}$ are vector functions on $\mathbb{R}^d$):

$$\langle\mathcal{R}\Phi,\Psi\rangle=2\int_{\mathbf{R}^{d+1}_+}\sum_{i,j=1}^d\frac{\partial^2\phi^h_j(x,t)}{\partial x_i\partial x_j}\psi^h_i(x,t)\,\mathrm{d}x\,\mathrm{d}t=2\int_{\mathbf{R}^{d+1}_+}\sum_{i,j=1}^d\frac{\partial\phi^h_j(x,t)}{\partial x_j}\frac{\partial\psi^h_i(x,t)}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t,$$

where we get the second equality because ϕ_j,ψ_i are smooth with compact support, and hence ϕ^h_j,ψ^h_i are Schwarz functions. Now, we only need to observe:

$$\int_{\mathbf{R}^{d+1}_+}\sum_{i,j=1}^d\frac{\partial\phi^h_j(x,t)}{\partial x_j}\frac{\partial\psi^h_i(x,t)}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t=\int_{\mathbf{R}^{d+1}_+}\sum_{i=1}^d\left(\sum_{j=1}^d\frac{\partial\phi^h_j(x,t)}{\partial x_i}\frac{\partial\psi^h_i(x,t)}{\partial x_j}\right)\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\mathbf{R}^{d+1}_+}trace\left[\left(\frac{\partial\phi^h_j(x,t)}{\partial x_i}\right)^d_{i,j=1}\left(\frac{\partial\psi^h_i(x,t)}{\partial x_j}\right)^d_{i,j=1}\right]\mathrm{d}x\,\mathrm{d}t$$

and that on the other hand, point-wisely:

$$\left|trace\left[\left(\frac{\partial\phi^h_j(x,t)}{\partial x_i}\right)^d_{i,j=1}\left(\frac{\partial\psi^h_i(x,t)}{\partial x_j}\right)^d_{i,j=1}\right]\right|\leqslant\left\|\left(\frac{\partial\phi^h_j(x,t)}{\partial x_i}\right)^d_{i,j=1}\right\|_2\left\|\left(\frac{\partial\psi^h_i(x,t)}{\partial x_j}\right)^d_{i,j=1}\right\|_2.\qquad\square$$

Issues of continuity of norms of Calderón–Zygmund operators have been coming up recently in connection with PDE with random coefficients. The recent preprint of Gloria and Otto – "An optimal variance estimate in stochastic homogenization of discrete elliptic equations" – uses the old continuity result of N. Meyer's (see [4]) for $\|T\|_p$ in p, at $p = 2$ to obtain rates of convergence in homogenization, where T stands for the 2-dimensional Ahlfors–Beurling operator, given by $T = R_1^2 - R_2^2 + 2iR_1R_2$, and in addition another recent preprint of Conlon and Spencer – "A strong central limit theorem for a class of random surfaces" (see [2]) – makes use of the fact that the $L^2(w)$ norm of a Calderón–Zygmund operator is close to its $L^2(\mathrm{d}x)$ norm, if the A_2 characteristic of the weight w is close to 1.

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