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Differential Geometry

A Note on surfaces with parallel mean curvature

Une Note sur des surfaces de courbure moyenne parallèle

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ABSTRACT

We use a Simons type equation in order to characterize complete non-minimal pmc surfaces with non-negative Gaussian curvature.

RÉSUMÉ

Dans cette Note, on étudie des immersions isométriques de surfaces complètes Σ^2 dans $M^n(c) \times \mathbb{R}$, ou $M^n(c)$ est une variété complète simplement connexe de courbure sectionnelle constante c. On classifie ces immersions, lorsque leur vecteur courbure moyenne est parallèle dans le fibré normal et leur courbure intrinsèque est positive ou nulle. L'outil principal est une différentielle quadratique holomorphe dont la partie sans trace satisfait l'équation de Codazzi.

1. The main result

Let $M^n(c)$ be a simply-connected n-dimensional manifold, with constant sectional curvature c, consider the product manifold $\bar{M} = M^n(c) \times \mathbb{R}$, and let Σ^2 be an immersed surface in $\bar{M}$.

Definition 1.1. The surface Σ^2 is called a *pmc surface* if its mean curvature vector H is parallel in the normal bundle. More precisely, $\nabla^\perp H = 0$, where the normal connection $\nabla^\perp$ is defined by the Weingarten equation

$$\bar{\nabla}_X V = -A_V X + \nabla^\perp_X V,$$

for any vector field X tangent to Σ^2 and any vector field V normal to the surface. Here $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ and A is the shape operator.

When the dimension of $\bar{M}$ is equal to 3, an immersed pmc surface in $\bar{M}$ is a surface with constant mean curvature (a *cmc surface*). U. Abresch and H. Rosenberg introduced in [1,2] a holomorphic differential on such surfaces and then completely classified those cmc surfaces on which it vanishes. In order to extend their results to the case of ambient spaces

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$\bar{M} = M^n(c) \times \mathbb{R}$, with $n \geqslant 2$, H. Alencar, M. do Carmo and R. Tribuzy defined in [3,4] a real quadratic form Q on pmc surfaces by

$$Q(X,Y) = 2\langle A_H X, Y\rangle - c\langle X,\xi\rangle\langle Y,\xi\rangle, \tag{1}$$

where ξ is the unit vector tangent to $\mathbb{R}$, and proved that its $(2,0)$-part (which for $n = 2$ is just the Abresch–Rosenberg differential) is holomorphic.

Using this quadratic form, we will prove the following:

Theorem 1.2. *Let $x : \Sigma^2 \to M^n(c) \times \mathbb{R}$, $c \neq 0$, be an isometrically immersed complete non-minimal pmc surface with non-negative Gaussian curvature. Then one of the following holds*:

(1) *the surface is flat*;
(2) *Σ^2 is a minimal surface of a totally umbilical hypersurface of $M^n(c)$*;
(3) *Σ^2 is a cmc surface in a 3-dimensional totally umbilical submanifold of $M^n(c)$*;
(4) *the surface lies in $M^4(c) \times \mathbb{R} \subset \mathbb{R}^6$ (endowed with the Lorentz metric), and there exists a plane P such that the level lines of the height function $p \to \langle x(p),\xi\rangle$ are curves lying in planes parallel to P.*

Remark 1.3. The same result was obtained by H. Alencar, M. do Carmo and R. Tribuzy in the case when $c < 0$ (Theorem 3 in [4]).

In order to prove Theorem 1.2 we will need the following Simons type equation obtained by S.-Y. Cheng and S.-T. Yau (Eq. (2.8) in [6]), which generalizes some previous results in [9–11]. Let N be an n-dimensional Riemannian manifold, and consider a symmetric operator S on N, that satisfies the Codazzi equation $(\nabla_X S)Y = (\nabla_Y S)X$, where ∇ is the Levi-Civita connection on the manifold. Then, we have

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 + \sum_{i=1}^{n} \lambda_i (\operatorname{trace} S)_{ii} + \frac{1}{2}\sum_{i,j=1}^{n} R_{ijij}(\lambda_i - \lambda_j)^2, \tag{2}$$

where λ_i, $1 \leqslant i \leqslant n$, are the eigenvalues of S, and R_{ijkl} are the components of the Riemannian curvature of N.

2. The proof of Theorem 1.2

Let us consider an operator S, defined on the surface Σ^2 by

$$S = 2A_H - c\langle T,\cdot\rangle T + \left(\frac{c}{2}|T|^2 - 2|H|^2\right)\mathrm{I}, \tag{3}$$

where T is the component of ξ tangent to the surface. When the ambient space is 3-dimensional this operator was introduced in [5]. We shall prove that $|S|^2$ is a bounded subharmonic function on the surface.

First, it is easy to see that

$$\langle SX, Y\rangle = Q(X,Y) - \frac{\operatorname{trace} Q}{2}\langle X, Y\rangle, \tag{4}$$

where Q is the quadratic form given by (1), which implies that S is symmetric and traceless. Another direct consequence of (4) is the following:

Lemma 2.1. *The $(2,0)$-part of Q vanishes on Σ^2 if and only if $S = 0$ on the surface.*

The following lemma is proved in [5]:

Lemma 2.2. *The operator S satisfies the Codazzi equation $(\nabla_X S)Y = (\nabla_Y S)X$, where ∇ is the Levi-Civita connection on the surface.*

From Lemma 2.2, Eq. (2) and the fact that $\operatorname{trace} S = 0$, we easily get

$$\frac{1}{2}\Delta|S|^2 = 2K|S|^2 + |\nabla S|^2, \tag{5}$$

where K is the Gaussian curvature of the surface.

Now, let us consider the local orthonormal frame field $\{E_3 = \frac{H}{|H|}, E_4, \dots, E_{n+1}\}$ in the normal bundle, and denote $A_\alpha = A_{E_\alpha}$. It follows that $\operatorname{trace} A_3 = 2|H|$ and $\operatorname{trace} A_\alpha = 0$, for all $\alpha > 3$.

From the definition (3) of S, we have, after a straightforward computation,

$$\det A_3 = \frac{1}{|H|^2}\det A_H = |H|^2 - \frac{1}{8|H|^2}|S|^2 - \frac{c^2}{16|H|^2}|T|^4 - \frac{c}{4|H|^2}\langle ST, T\rangle,$$

and then, by using the equation of Gauss of Σ^2 in $\bar{M}$,

$$R(X,Y)Z = c\big\{\langle Y,Z\rangle X - \langle X,Z\rangle Y - \langle Y,T\rangle\langle Z,T\rangle X + \langle X,T\rangle\langle Z,T\rangle Y + \langle X,Z\rangle\langle Y,T\rangle T - \langle Y,Z\rangle\langle X,T\rangle T\big\}$$
$$+ \sum_{\alpha=3}^{n+1}\big\{\langle A_\alpha Y, Z\rangle A_\alpha X - \langle A_\alpha X, Z\rangle A_\alpha Y\big\}.$$

The Gaussian curvature can be written as

$$K = c\big(1-|T|^2\big) + |H|^2 - \frac{1}{8|H|^2}|S|^2 - \frac{c^2}{16|H|^2}|T|^4 - \frac{c}{4|H|^2}\langle ST, T\rangle + \sum_{\alpha>3}\det A_\alpha. \tag{6}$$

Since $\operatorname{trace} A_\alpha = 0$, it follows that $\det A_\alpha \leqslant 0$, for all $\alpha > 3$. Therefore, as $K \geqslant 0$, we get

$$-\frac{1}{8|H|^2}|S|^2 - \frac{c}{4|H|^2}\langle ST, T\rangle - \frac{c^2}{16|H|^2}|T|^4 + c\big(1-|T|^2\big) + |H|^2 \geqslant 0.$$

From $|\langle ST, T\rangle| \leqslant \frac{1}{\sqrt{2}}|T||S|$ it results that $-\frac{c}{4|H|^2}\langle ST, T\rangle \leqslant \frac{|c|}{4\sqrt{2}|H|^2}|S|$, which implies

$$-\frac{1}{8|H|^2}|S|^2 + \frac{|c|}{4\sqrt{2}|H|^2}|S| + c\big(1-|T|^2\big) + |H|^2 \geqslant 0.$$

Next, we shall consider two cases as $c<0$ or $c>0$, and will prove that, in both situations, $|S|$ is bounded from above. If $c<0$ we have

$$-\frac{1}{8|H|^2}|S|^2 - \frac{c}{4\sqrt{2}|H|^2}|S| + |H|^2 \geqslant 0$$

and then $|S| \leqslant \frac{\sqrt{c^2+|H|^2}-c}{\sqrt{2}}$.

When $c>0$ it follows that

$$-\frac{1}{8|H|^2}|S|^2 + \frac{c}{4\sqrt{2}|H|^2}|S| + c + |H|^2 \geqslant 0,$$

which is equivalent to $|S| \leqslant \frac{\sqrt{c^2+16c|H|^2+16|H|^2}+c}{\sqrt{2}}$.

As the surface is complete and has non-negative Gaussian curvature, it follows, from a result of A. Huber in [8], that Σ^2 is a parabolic space. From the above calculation and (5), we get that $|S|^2$ is a bounded subharmonic function and it follows that $|S|$ is a constant. Again using Eq. (5), one concludes that $K=0$ or $S=0$. From Lemma 2.1, we see that, when Σ^2 is not flat, the $(2,0)$-part of the quadratic form Q vanishes on the surface, and then we obtain the last three items of our Theorem exactly as in the proofs of Theorem 2 and Theorem 3 in [4].

Remark 2.3. M. Batista characterized some cmc surfaces in $M^2(c)\times\mathbb{R}$, under some assumptions on their mean curvature and on $|S|$. Since these assumptions imply that these surfaces have non-negative Gaussian curvature (this can be easily verified by using (6) and the fact that $|ST|^2 = \frac{1}{2}|T|^2|S|^2$. We remark the converse is not necessarily true), we can see that Theorem 3.1 in [7] generalizes his results (Theorem 1.2 and Theorem 1.3 in [5]).

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