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A new local reduced basis discontinuous Galerkin approach for heterogeneous multiscale problems

Une nouvelle approche par bases réduites locale d'une méthode de Galerkin discontinue pour des problèmes hétérogènes multi-échelles

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ABSTRACT

Inspired by the reduced basis approach and modern numerical multiscale methods, we present a new framework for an efficient treatment of heterogeneous multiscale problems. The new approach is based on the idea of considering heterogeneous multiscale problems as parametrized partial differential equations where the parameters are smooth functions. We then construct, in an offline phase, a suitable localized reduced basis that is used in an online phase to efficiently compute approximations of the multiscale problem by means of a discontinuous Galerkin method on a coarse grid. We present our approach for elliptic multiscale problems and discuss an a posteriori error estimate that can be used in the construction process of the localized reduced basis. Numerical experiments are given to demonstrate the efficiency of the new approach.

RÉSUMÉ

Inspiré par l'approche des bases réduites et les méthodes numériques modernes pour des problèmes multi-échelles, nous présentons un nouveau traitement efficace des problèmes hétérogènes multi-échelles. La nouvelle approche repose sur l'idée de considérer des problèmes hétérogènes multi-échelles comme des équations différentielles partielles paramétrisées, où les paramètres sont des fonctions lisses. Nous construisons alors dans une phase « offline » une base réduite localisée appropriée, utilisée dans une phase « online » pour calculer efficacement des approximations du problème multi-échelle par une méthode Galerkin discontinue sur un maillage grossier. Nous présentons notre nouvelle approche pour des problèmes elliptiques multi-échelles et discutons une estimation d'erreur à posteriori utilisée lors de la construction de la base réduite localisée. Des expériences numériques sont exposées pour démontrer la efficacité de la nouvelle approche.

Version française abrégée

Dans les derniers années, les problèmes multi-échelles ont retenu une attention particulière dans la recherche mathématique appliquée. Un traitement numérique de ces problèmes est généralement coûteux et nécessite donc le développement

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de schémas de discrétisation efficaces. En outre, ces problèmes souvent être résolus à plusieurs reprises pour de nombreux paramètres. Des exemples de telles situations sont la conception, le contrôle, l'optimisation, les problèmes inverses, l'analyse d'incertitude et l'échantillonnage statistique. L'objectif principal de cette contribution est de rassembler des idées de l'approche des bases réduites et des méthodes numériques multi-échelles pour obtenir un nouveau cadre pour gérer efficacement les problèmes complexes dépendant de plusieurs paramètres. Nous présentons notre approche pour un modèle associé à la mécanique des fluides diphasiques. En particulier, nous considérons l'Éq. (1). Ici λ désigne une fonction lisse sur une échelle macro, k^ϵ est censé varier sur une échelle micro, indiquée par $\epsilon \ll 1$. La mobilité totale du fluide $\lambda = \lambda(s(x,t))$ dépend de la saturation inconnue de l'une des phases.

Le cadre général est résumé comme suit : Trouver un petit nombre de fonctions représentantes p_i, $i = 1, \ldots, N$, tel que pour toutes les fonctions de paramètre λ possibles, il existe une fonction S lisse, éventuellement non-linéaire, avec $\|p(\lambda(x); x) - S(p_1, \ldots, p_N)(x)\| \leqslant \mathrm{TOL}$, où $\|\cdot\|$ désigne une norme appropriée et TOL est une tolérance d'erreur donnée. La méthode suivante nous permet d'attendre ce but. Par définition (2) la forme bilinéaire b et la forme linéaire l que nous utilisons pour établir la formulation éléments finis (3) dans un maillage fin $\mathcal{T}_h$. Avec les solutions $p_h^{\lambda_i}$ de ce schéma pour paramètres differents λ_i nous construisons une base réduite Φ_F pour chaque élément F d'un maillage grossier $\mathcal{Z}_H$ et obtenons l'espace W_N, voir (4). Nous définissons un schéma grossier (7) dans ce espace avec l'utilisation d'une forme bilinéaire Galerkin discontinue (5) et le côté droit (6). Ensuite, par le Théorème 3.1, nous introduisons une estimation de l'erreur entre la solution faible p^λ (voir (2)) et la solution grossière p_N^λ (voir (7)).

Les résultats numériques démontrent les possibilités de notre approche. Nous utilisons les fonctions données en (10) avec $N_S = 3$ et $(\mu_1, \mu_2, \mu_3) \in [0.1, 1.0] \times [0.1, 0.5] \times [0.1, 0.101]$ sur $\Omega = [0, 10]^2$, $f = 0$. Les valeurs sur le bord du domaine sont données par $g_D(x) = \sum_{n=1}^{N_\lambda} \mu_n^\lambda (11 - x_1)$, où les μ_i^λ sont les paramètres de λ. Ici, en vertu de la forme de λ, N_λ est une fonction de N_S et les μ_n^λ sont des fonctions des $\mu_1, \ldots, \mu_{N_S}$. Le Tableau 1 nous montre quelles réduction d'ordre sont possible. À l'aide du maillage indiqué en Fig. 1 nous réduisons le temps nécessaire pour une simulation par un facteur, dépendant de la grandeur du maillage fin, de 4 à 40 (temps pour une simulation fine par rapport au temps pour une simulation grossière incluant la reconstruction d'une fonction sur un maillage fin), tandis que l'erreur relative moyenne additionnelle $\|p_h^{\lambda_i} - p_N^{\lambda_i}\|_{L^2} / \|p_h^{\lambda_i}\|$ pour 125 fonctions λ_i est inférieure à 8×10^{-4}. En outre, dans la Fig. 2 nous voyons que les comportements à l'échelle macro aussi bien que les détails à l'échelle micro sont bien représentés.

1. Introduction

In recent years, multi-physics and multiscale problems have become a particular focus of applied mathematical research. A numerical treatment of such problems is usually very time consuming and thus requires the development of efficient discretization schemes that are often realized on large parallel computing environments. In addition, these problems often need to be solved repeatedly for many varying parameters, introducing a curse of dimensionality when the solution is also viewed as a function of these parameters. Examples for such situations include design, control, optimization, inverse problems, uncertainty analysis and statistical sampling. There has been a tremendous effort in developing efficient approaches to deal with such problems in various mathematical communities. In particular, relevant research areas are system-theoretic model order reduction for dynamical systems, proper orthogonal decomposition, reduced basis methods for parametrized PDEs, and numerical multiscale methods.

The major goal of this contribution is to bring together ideas from the reduced basis approach and numerical multiscale methods to derive a new framework to efficiently handle complex parameter dependent multiscale problems. We present our new approach for a model problem associated with two phase flow in porous media. In particular, we consider the equation

$$-\nabla \cdot \big(\lambda(x) k^\epsilon(x) \nabla p\big(\lambda(x); x\big)\big) = f(x). \tag{1}$$

Here λ denotes a coefficient function that is assumed to be smooth on a coarse scale, while k^ϵ is supposed to vary on a fine scale, indicated by $\epsilon \ll 1$. In the context of two phase flow in porous media, this model problem is part of the so-called global pressure formulation. In this situation, $\lambda = \lambda(s(x,t))$ denotes the total mobility of the fluid and depends on the unknown volume saturation s of one of the phases.

The overall framework is summarized as follows. Find a small number of representative fields p_i, $i = 1, \ldots, N$, such that for all admissible parameter functions λ there exists a smooth, possibly non-linear mapping S with $\|p(\lambda(x); x) - S(p_1, \ldots, p_N)(x)\| \leqslant \mathrm{TOL}$, where $\|\cdot\|$ denotes a suitable norm and TOL is a given error tolerance. The objective for a resulting numerical method is then to construct such optimal representative fields and to derive efficient schemes to compute an approximation of $S(p_1, \ldots, p_N)$. In addition, a posteriori error estimators are to be derived to ensure reliability and efficiency of the resulting numerical schemes. In the classical reduced basis approach the mapping S induces a linear combination $\sum_{i=1}^N a_i p_i$ with suitably chosen coefficients a_i, $i = 1, \ldots, N$. As the parameter in our case is a function itself, the principal ansatz in this contribution will be a generalization of the form $S(p_1, \ldots, p_N)(x) = \sum_{i=1}^N a_i(x) p_i(x)$ where the parameter functions a_i, $i = 1, \ldots, N$, are to be determined by a macroscopic discontinuous Galerkin scheme. On the one hand, this approach follows the ideas of the so-called multiscale finite element approach using limited global information [1], while at the same time it comprises some similarities with the so-called reduced basis element approach [7,6,3]. In contrast to the

cited approaches, however, we make use of the classical penalization approach in discontinuous Galerkin methods to couple our localized representative fields, instead of using mixed methods or mortar techniques, respectively.

The rest of the Note is outlined as follows. In Section 2, we introduce the model problem and define our new reduced basis multiscale discontinuous Galerkin method. An a posteriori error estimate is given in Section 3 where we also sketch an adaptive algorithm that is used to construct efficient localized representative fields. We conclude with a numerical experiment that demonstrates the efficiency of our new approach.

2. A new reduced basis multiscale discontinuous Galerkin method

Our new approach consists of two steps. In an offline phase we compute standard finite element approximations of our fine scale problem for particular choices of the parameter function and possibly also other parameters. From these solutions that we will call snapshots or representative fields in the following, we construct localized basis functions defined on a coarse macro grid. This is done by restricting the snapshots to macro elements and further compressing them locally by applying a proper orthogonal decomposition (POD, [4]). The resulting reduced local bases will then serve in a second online step as the discontinuous ansatz and test space of a macroscopic interior penalty discontinuous Galerkin scheme. To set up our framework, we start with a proper definition of our model problem.

Definition 2.1 *(Weak formulation).* Let Ω denote an open bounded set in $\mathbb{R}^d$, $d \in \{2, 3\}$, $g_D \in H^{\frac{1}{2}}(\partial\Omega)$, and $\overline{g_D} \in H^1(\Omega)$ such that $\overline{g_D}|_{\partial\Omega} = g_D$. Let $f \in L^2(\Omega)$, $k^\epsilon \in L^\infty(\Omega)$ be given where ϵ indicates the fine scale fluctuations associated with k^ϵ. Define $b(\lambda; v, w) = \int_\Omega \lambda k^\epsilon \nabla v \cdot \nabla w$, and $l(v) = \int_\Omega f v - \int_\Omega \lambda k \nabla \overline{g_D} \cdot \nabla v$. For a given smooth parameter function λ, we then look for a solution $p^\lambda = \overline{g_D} + p_0^\lambda$, where $p_0^\lambda \in H_0^1(\Omega)$ satisfies

$$b\big(\lambda, p_0^\lambda, v\big) = l(v) \quad \forall v \in H_0^1(\Omega). \tag{2}$$

2.1. Fine scale finite element approximation

Definition 2.2 *(Nested coarse & fine grids and function spaces).* Let $\mathcal{T}_h$ be a conforming triangulation of the domain Ω and $\mathcal{Z}_H$ a coarse grid such that $\mathcal{T}_h$ is a nested refinement of $\mathcal{Z}_H$. Denote the inner faces of $\mathcal{T}_h$ by Γ_I, the boundary ones by Γ_B and the set of all faces by Γ. Let Ξ_I, Ξ_B and Ξ denote the respective quantities in $\mathcal{Z}_H$. With each $e \in \Gamma$ we associate a unique unit normal vector $\boldsymbol{n}_e$. Furthermore, let $h_E = \operatorname{diam}(E)$ be the diameter of $E \in \mathcal{T}_h$ and $h = \max_{E\in\mathcal{T}_h} h_E$ be the mesh size of $\mathcal{T}_h$.

On $\mathcal{T}_h$ we introduce the discrete function spaces $V_{h,k} = \{v \in L^2(\Omega) \mid v|_E \in \mathbb{P}_k(E)\ \forall E \in \mathcal{T}_h\}$, $S_{h,k} = H_0^1(\Omega) \cap V_{h,k}$, $S_{h,k}^0 = \{v \in S_{h,k} \mid v|_{\partial\Omega} = 0\}$.

Definition 2.3 *(Fine scale finite element approximation).* Let the parameter function λ be given. We then define the fine scale finite element solution $p_h^\lambda = \overline{g_D}_h + p_{h,0}^\lambda$ with $\overline{g_D}_h \in S_{h,1}$ and $p_{h,0}^\lambda \in S_{h,1}^0$ defined through

$$b\big(\lambda; p_{h,0}^\lambda, v_h\big) = l(v_h) \quad \forall v_h \in S_{h,1}^0. \tag{3}$$

We use this finite element approximation to compute in an offline phase representative fields $p_h^{\lambda_i}$, $i \in I \subset \mathbb{N}$, for a suitable choice of parameter functions λ_i.

2.2. Coarse scale discontinuous Galerkin scheme

Let us now suppose that we are given a localized reduced basis on each coarse grid cell $F \in \mathcal{Z}_H$, i.e. $\Phi_F := \{\varphi_F^1, \ldots, \varphi_F^{N_F}\} \subset S_{h,k}(F)$ where $S_{h,k}(F)$ denotes the restriction of $S_{h,k}$ to the coarse grid cell F. We then define the coarse scale reduced broken space W_N by

$$W_N = \big\{v_N \in L^2(\Omega) \;\big|\; v_N|_F \in \operatorname{span}(\Phi_F)\ \forall F \in \mathcal{Z}_H\big\}, \tag{4}$$

where $N = \dim(W_N) = \sum_{F\in\mathcal{Z}_H} N_F$ denotes the degrees of freedom of this macroscopic discontinuous reduced space. As the space W_N is discontinuous across the faces of the macro mesh, we define for a face $e \in \Xi_I$ with unit normal vector $\boldsymbol{n}_e$ oriented from neighboring elements F_1 to F_2 the average and jump of $v_N \in W_N$ on e as $\{v\} = \frac{1}{2}(v|_{F_1} + v|_{F_2})$, $[v] = v|_{F_1} - v|_{F_2}$. For a face $e \in \Xi_B$ we define the respective quantities on e as $\{v\} = v$, $[v] = v$.

Definition 2.4 *(Reduced basis multiscale discontinuous Galerkin scheme).* Let us denote the bilinear form

$$B_{\mathrm{DG}}(\lambda; v, w) = \sum_{F\in\mathcal{Z}_H} \int_F \lambda k \nabla v \cdot \nabla w - \sum_{e\in\Xi} \int_e \{\lambda k \nabla v \cdot \boldsymbol{n}_e\}[w] - \sum_{e\in\Xi} \int_e \{\lambda k \nabla w \cdot \boldsymbol{n}_e\}[v] + J_{\sigma,\beta}(v, w), \tag{5}$$

where the penalty contribution is given through $J_{\sigma,\beta}(v, w) = \sum_{e\in\Xi} \frac{\sigma}{|e|^\beta} \int_e [v][w]$ with stabilization parameters $\sigma > 0$, $\beta > 0$. Additionally, set

$$L(\lambda; v) = \sum_{F\in\mathcal{Z}_H}\int_F f v + \sum_{e\in\Xi_B}\int_e \left(\frac{\sigma}{|e|^\beta} v - \lambda k \nabla v\cdot \boldsymbol{n}\right) g_D. \tag{6}$$

For given parameter function λ, we then define the reduced basis multiscale discontinuous Galerkin approximation $p_N^\lambda \in W_N$ as a solution of

$$B_{\mathrm{DG}}\big(\lambda; p_N^\lambda, v_N\big) = L(\lambda; v_N) \quad \forall v_N \in W_N. \tag{7}$$

3. A posteriori error estimate and adaptive algorithm

We now state an a posteriori error estimator for the reduced basis multiscale discontinuous Galerkin scheme that will also help us to efficiently construct the localized reduced space W_N and – as it is offline–online decomposable – provide an efficient online control of the approximation error.

For simplicity of the exposition, but without loss of generality, we will restrict the parameter functions λ and k^ϵ in the following theorem to fulfill $(\lambda k^\epsilon)|_E \in \mathbb{P}_1$ for all $E \in \mathcal{T}_h$ and $(\lambda k^\epsilon) \in C^0(\Omega)$. Furthermore, we introduce a linear post processing operator $\mathcal{R}_{1,2} : V_{h,1} \to V_{h,2} \cap H^1(\mathcal{Z}_H)$, $H^1(\mathcal{Z}_H) := \{v \in L^2(\Omega) \mid v|_F \in H^1(F)\ \forall F \in \mathcal{Z}_H\}$. The concrete choice of this operator will not be discussed here. For the theoretical results it suffices to know that for any piecewise linear function v_h, $\mathcal{R}_{1,2}(v_h)$ is a piecewise polynomial of higher order that is continuous inside the macro elements $F \in \mathcal{Z}_H$.

Theorem 3.1 *(A posteriori error estimate). For a given parameter function λ, let p^λ and p_N^λ be the weak* (2) *and discontinuous Galerkin solution* (7)*, respectively. Then, the following error estimate holds*

$$\big\|p^\lambda - p_N^\lambda\big\|_{0,\Omega} \leqslant \big\|\mathcal{R}_{1,2}\big(p_N^\lambda\big) - p_N^\lambda\big\|_{0,\Omega} + \sum_{F\in\mathcal{Z}_H} \eta_1^F\big(\mathcal{R}_{1,2}\big(p_N^\lambda\big)\big) + \sum_{e\in\Gamma_I} \eta_2^e\big(\mathcal{R}_{1,2}\big(p_N^\lambda\big)\big) + \sum_{e\in\Xi_B} \eta_3^e\big(\mathcal{R}_{1,2}\big(p_N^\lambda\big)\big), \tag{8}$$

where the indicators are given for $\xi \in V_{h,2} \cap H^1(\mathcal{Z}_H)$ as

$$\begin{aligned}
\eta_1^F(\xi) &= \frac{C_o^2}{k_1}\big\|f + \nabla\cdot(\lambda k\nabla\xi)\big\|_{0,F} + C_r\left(\frac{C_o k_2}{k_1} + h_e\right)\sum_{e\subset\partial F}\big\|\boldsymbol{r}_e(\xi)\big\|_{0,\Omega},\\
\eta_2^e(\xi) &= (C_o + h_e)\frac{C_r C_o}{k_1}\big\|\boldsymbol{r}_e(\lambda k\nabla\xi\cdot\boldsymbol{n})\big\|_{0,\Omega}, \qquad \eta_3^e(\xi) = C_r\left(\frac{C_o k_2}{k_1} + h_e\right)\big\|\boldsymbol{r}_e(\xi - g_D)\big\|_{0,\Omega}.
\end{aligned} \tag{9}$$

Here, $\boldsymbol{r}_e : L^2(e) \to [V_{h,2}]^2$ denotes a lifting operator (cf. [2]*) for the jumps across the faces $e \in \Gamma$ or $e \in \Xi$ defined through $\int_\Omega \boldsymbol{r}_e(w)\cdot\boldsymbol{\tau}_h = -\int_e [w]\{\boldsymbol{\tau}_h\cdot\boldsymbol{n}\}$ for all $\boldsymbol{\tau}_h \in [V_{h,2}]^2$ and C_o and C_r are computable constants only depending on the domain.*

Proof. The proof is based on a general a posteriori approach via duality techniques for consistent and adjoint consistent local discontinuous Galerkin schemes in the sense of Arnold et al. [2]. The approach is sketched in the lecture notes [8] and will be elaborated in the current context in a forthcoming paper. □

3.1. Offline/online decomposition and adaptive construction of the localized reduced space

In the offline step, the reduced DG space W_N is constructed and all integrals of the parameter independent contributions in the definition of the macroscopic discontinuous Galerkin method and of the a posteriori error estimator are precomputed. This step is performed only once. In the online step, solutions of the reduced problem are computed for given parameter functions λ with the reduced basis multiscale DG approximation. This step has low computational complexity that only depends on the reduced degrees of freedom N and is usually performed multiple times for different choices of the parameter function.

We now describe in more detail how the basis functions of W_N are generated in the offline stage. We therefore introduce a so-called training set $\mathcal{M}_{\mathrm{train}}$ that consists of a finite set of parameter functions λ_i, $i \in I_{\mathrm{train}}$. Given an error tolerance Δ, a maximum basis size $N_{\max}$ and a POD-tolerance Δ_{POD}, the basis Φ of W_N is then generated as follows:

1. Set $\tilde{\Phi}_{-1}, \tilde{\Phi}_{-1,F} := \emptyset$ for all $F \in \mathcal{Z}_H$ and choose a parameter function $\lambda_0 \in \mathcal{M}_{\mathrm{train}}$ for the construction of an initial basis.
2. Let a basis $\tilde{\Phi}_{k-1} = \bigcup_{F\in\mathcal{Z}_H} \tilde{\Phi}_{k-1,F}$ and a parameter function λ_k be given. Perform a detailed simulation to obtain $p_h^{\lambda_k}$ and define a preliminary basis extension $\tilde{\varphi}_F$, $F \in \mathcal{Z}_H$, by $\tilde{\varphi}_F := p_h^{\lambda_k}|_F, \forall F \in \mathcal{Z}_H$. Add $\tilde{\varphi}_F$, $F \in \mathcal{Z}_H$, to the basis $\tilde{\Phi}_{k-1,F}$ and obtain $\tilde{\Phi}_{k,F}$ and $\tilde{\Phi}_k = \bigcup_{F\in\mathcal{Z}_H} \tilde{\Phi}_{k,F}$.
3. Compute the offline-parts of the DG scheme and of the error estimator for the current basis $\tilde{\Phi}_k$.
4. Compute the reduced solutions p_N^λ for all $\lambda \in \mathcal{M}_{\mathrm{train}}$ using the current basis. Then evaluate an error estimator (e.g. (8)) for all these solutions and find the parameter function $\lambda_{k+1} \in \mathcal{M}_{\mathrm{train}}$ that maximizes the error bound.

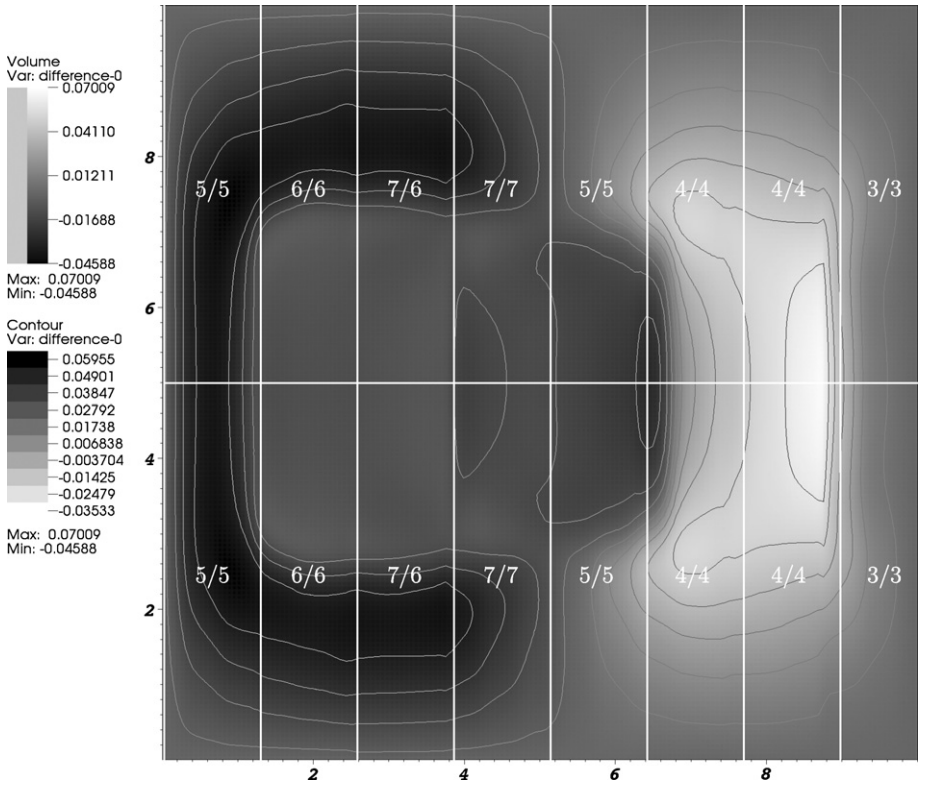

Fig. 1. Difference between fine scale and reduced solution from Fig. 2 and coarse triangulation (white) with number of reduced basis functions $|\Phi_F|$ for each coarse cell F for the fine triangulation with 2048 and 32 768 cells.

5. **If** $N < N_{\max}$ and if the error bound for the reduced solution $p_N^{\lambda_{k+1}}$ is larger than Δ, continue with Step 2 with λ_{k+1} from Step 4.
   Else Apply the POD with accuracy Δ_{POD} to $\tilde{\Phi}_{k,F}$ on each $F \in \mathcal{Z}_H$ and obtain the reduced orthogonalized local bases Φ_F and the global basis $\Phi = \bigcup_{F\in\mathcal{Z}_H} \Phi_F$ with $N := |\Phi|$.

In Step 5 we control the sizes of the new bases Φ_F by specifying the mean quadratic POD projection error which is given by the sum of all eigenvalues of the covariance matrix that are neglected when building the new basis. This procedure may lead to different numbers of base functions on each coarse cell, reflecting high or low local solution variability. As the error estimator measures the difference between exact and reduced solution, the algorithm may stall once a certain richness of the reduced space is reached. Also, after the POD projection in Step 5, the maximum estimated error may be slightly larger than Δ. Both problems were addressed in [5] and will be dealt with in a forthcoming paper.

4. Numerical experiments

In this section we demonstrate the multiscale and model order reduction capabilities of our approach. Using the algorithm from Section 3.1 we build a reduced basis for model problem (1) in two spatial dimensions. Here we set

$$k^\varepsilon(x) = \frac{2}{3}(1+x_1)\left(1+\cos^2\left(2\pi\frac{x_1}{\varepsilon}\right)\right), \qquad \lambda(x) = \frac{1}{\eta_o} - \frac{2}{\eta_o}S(x) + \frac{\eta_o+\eta_w^2}{\eta_w\eta_o}\sum_{m,n=1}^{N_S}\mu_n\mu_m S_n(x)S_m(x),$$

$$S(x) = \sum_{n=1}^{N_S}\mu_n S_n(x), \quad S_n(x) = 0.1\left(\arctan\left(-(2n-0.25N_S x_1)^6 - 0.01(x_2-5)^6\right) + \frac{5\pi}{6}\right)^3 \tag{10}$$

with $\varepsilon = 1$, $N_S = 3$ and $(\mu_1,\mu_2,\mu_3) \in [0.1,1.0]\times[0.1,0.5]\times[0.1,0.101]$ on $\Omega = [0,10]^2$, $f = 0$. We use the Dirichlet boundary values $g_D(x) = \sum_{n=1}^{N_\lambda}\mu_n^\lambda(11-x_1)$, where N_λ is a function of N_S and the μ_n^λ are functions of $\mu_1,\dots,\mu_{N_S}$.

Using the coarse grid indicated in Fig. 1 and a fine triangulation with 2048 cells we are able to reduce the mean solution time by a factor of 4 (time for fine scale solution to time for reduced solution including reconstruction) while introducing an additional mean relative error (mean of $\|p_h^\lambda - p_N^\lambda\|_{L^2}/\|p_h^\lambda\|_{L^2}$ over 125 test parameter functions λ) of only 4.74×10^{-4}. Refining the fine grid $\mathcal{T}_h$ to 32 768 cells we even get a factor of 40 with an additional mean error of only 7.59×10^{-4} (cf. Table 1). As typical in RB methods, the online efficiency comes by the price of an offline-phase that is several orders more expensive.

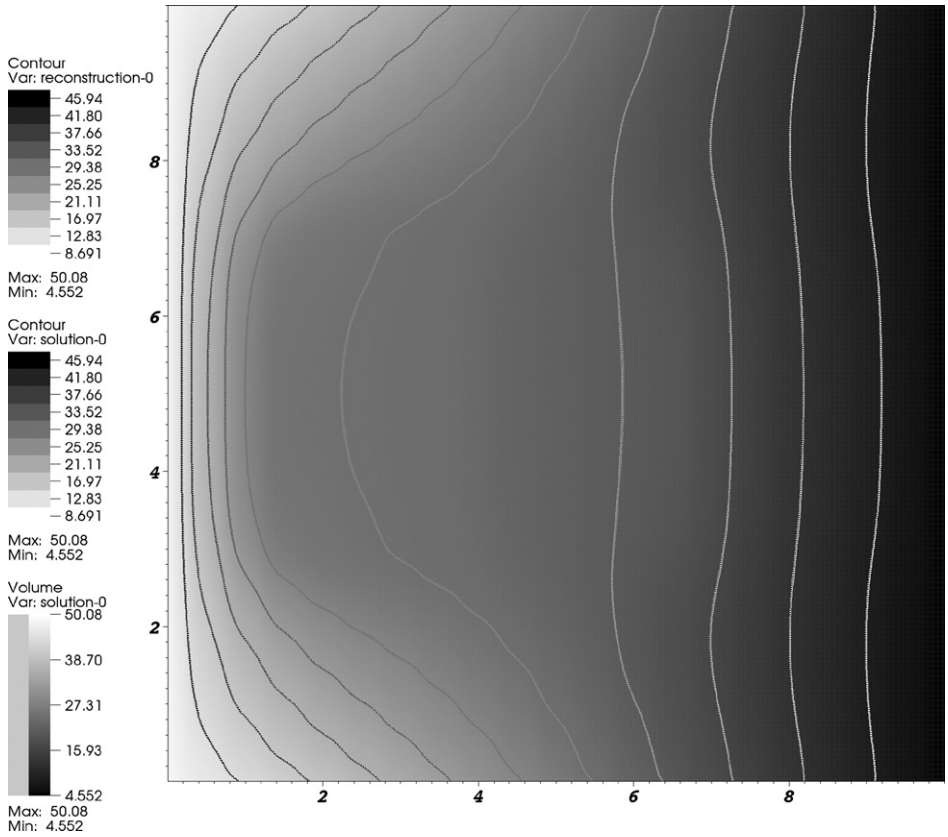

Fig. 2. Contour plots of fine scale solution (solid lines) and reconstructed reduced solution (dotted lines) to Eq. (1) for $\mu_1 = 0.85$, $\mu_2 = 0.5$, $\mu_3 = 0.1$, volume representation of reconstructed reduced solution. Here: $|\mathcal{T}_h| = 32\,768$.

Table 1
Runtimes for the basis generation algorithm (t_{basisgen}), high and low dimensional algorithms ($t_{\text{high dim}}$ and $t_{\text{low dim}}$) and the reconstruction (t_{recons}) and error values ($\epsilon = \|p_h^\lambda - p_N^\lambda\|_{L^2} / \|p_h^\lambda\|_{L^2}$) for different grid sizes. The values in columns 3–7 are mean values over 125 test parameter functions λ.

$\lvert\mathcal{T}_h\rvert$	N	$t_{\text{high dim}}$ (s)	$t_{\text{low dim}}$ (ms)	t_{recons} (ms)	Factor	ϵ
2048	82	0.19	8.54	36.78	4	4.74×10^{-4}
8192	80	2.59	9.93	151.4	16	6.44×10^{-4}
32 768	80	22.58	12.24	545.3	40	7.59×10^{-4}

In addition to those quite convincing model order reduction capabilities, we demonstrate the multiscale capabilities of our approach in Fig. 2 where we see, that the behavior on the fine scale (oscillations due to oscillating k^ε) as well as the behavior on the coarse scale is captured correctly.

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