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An intrinsic approach and a notion of polyconvexity for nonlinearly elastic plates

Une approche intrinsèque et une notion de polyconvexité pour les plaques non linéairement élastiques

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ABSTRACT

Let ω be a domain in $\mathbb{R}^2$. The classical approach to the Neumann problem for a nonlinearly elastic plate consists in seeking a displacement field $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ that minimizes a non-quadratic functional over $\boldsymbol{V}(\omega)$. We show that this problem can be recast as a minimization problem in terms of the new unknowns $E_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) \in L^2(\omega)$ and $F_{\alpha\beta} = \partial_{\alpha\beta}\eta_3 \in L^2(\omega)$ and that this problem has a solution in a manifold of symmetric matrices $\boldsymbol{E} = (E_{\alpha\beta})$ and $\boldsymbol{F} = (F_{\alpha\beta})$ whose components $E_{\alpha\beta} \in L^2(\omega)$ and $F_{\alpha\beta} \in L^2(\omega)$ satisfy nonlinear compatibility conditions of Saint-Venant type. We also show that such an "intrinsic approach" naturally leads to a new definition of polyconvexity.

RÉSUMÉ

Soit ω un domaine de $\mathbb{R}^2$. L'approche classique du problème de Neumann pour une plaque non linéairement élastique consiste à chercher un champ de déplacements $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ qui minimise une fonctionnelle non quadratique sur $\boldsymbol{V}(\omega)$. Nous montrons que ce problème peut être ré-écrit comme un problème de minimisation en termes des nouvelles inconnues $E_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) \in L^2(\omega)$ et $F_{\alpha\beta} = \partial_{\alpha\beta}\eta_3 \in L^2(\omega)$ et que ce problème a une solution dans une variété de matrices symétriques $\boldsymbol{E} = (E_{\alpha\beta})$ et $\boldsymbol{F} = (F_{\alpha\beta})$ dont les composantes $E_{\alpha\beta} \in L^2(\omega)$ et $F_{\alpha\beta} \in L^2(\omega)$ satisfont des conditions non linéaires de compatibilité du type de Saint-Venant. Nous montrons également qu'une telle « approche intrinsèque » conduit naturellement à une nouvelle définition de polyconvexité.

1. The classical approach to the Neumann problem for a nonlinearly elastic plate

This Note is a sequel to the Note [3], to which we refer for the notations and definitions not recalled here. Let ω be a domain in $\mathbb{R}^2$ and let

$$\boldsymbol{V}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega). \tag{1.1}$$

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In the classical Kirchhoff–von Kármán–Love theory, the *Neumann problem for a nonlinearly elastic plate* with middle surface $\bar{\omega}$ consists in finding a *vector field* $\boldsymbol{\zeta} = (\zeta_i) \in \boldsymbol{V}(\omega)$ (the displacement vector field of $\bar{\omega}$) that minimizes over the space $\boldsymbol{V}(\omega)$ the functional $J : \boldsymbol{V}(\omega) \to \mathbb{R}$ defined for each $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega)$ by

$$J(\boldsymbol{\eta}) := \frac{1}{2}\int_\omega \left\{ \frac{\varepsilon}{4} a_{\alpha\beta\sigma\tau}(\partial_\sigma\eta_\tau + \partial_\tau\eta_\sigma + \partial_\sigma\eta_3\partial_\tau\eta_3)(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3) + \frac{\varepsilon^3}{3} a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\eta_3\partial_{\alpha\beta}\eta_3 \right\} \mathrm{d}\omega - L(\boldsymbol{\eta}), \tag{1.2}$$

where

$$L(\boldsymbol{\eta}) := \int_\omega p_i\eta_i \,\mathrm{d}\omega - \int_\omega q_\alpha\partial_\alpha\eta_3 \,\mathrm{d}\omega. \tag{1.3}$$

In (1.2), $\varepsilon > 0$ denotes half of the thickness of the plate, and the constants $a_{\alpha\beta\sigma\tau}$, which denote the components of the *two-dimensional elasticity tensor* of the plate, satisfy

$$a_{\alpha\beta\sigma\tau}t_{\sigma\tau}t_{\alpha\beta} \geqslant 4\mu\sum_{\alpha,\beta}|t_{\alpha\beta}|^2 \quad \text{for all } (t_{\alpha\beta}) \in \mathbb{S}^2, \tag{1.4}$$

for some constant $\mu > 0$ (one of the Lamé constants of the constituting material of the plate, assumed to be homogeneous and isotropic; the reference configuration $\bar{\omega} \times [-\varepsilon, \varepsilon]$ of the plate is assumed to be a natural state). In the linear form $L : \boldsymbol{V}(\omega) \to \mathbb{R}$ defined by (1.2), the functions $p_i \in L^2(\omega)$ and $q_\alpha \in L^2(\omega)$ are given (they represent the resultants of the forces that are applied to the plate).

The objective of this Note is to establish the *existence* of a solution to this minimization problem, by means of a reformulation of this minimization problem in terms of the *new unknowns*

$$E_{\alpha\beta} := \frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3) \in L^2(\omega) \quad \text{and} \quad F_{\alpha\beta} := \partial_{\alpha\beta}\eta_3 \in L^2(\omega), \quad \alpha, \beta = 1, 2, \tag{1.5}$$

i.e., by means of an *intrinsic approach*.

Complete proofs will be found in [4].

2. Necessary conditions for the existence of a minimizer

If the plate is *linearly elastic*, i.e., if the nonlinear functions $E_{\alpha\beta}$ defined in (1.4) are replaced by their linear parts

$$e_{\alpha\beta} := \frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha), \quad \alpha, \beta = 1, 2, \tag{2.1}$$

the functional J of (1.2) is replaced by a *quadratic functional*. In this case, it is clear that a necessary (and in effect sufficient) condition for the existence of a minimizer of this quadratic functional over the space $\boldsymbol{V}(\omega)$ of (1.1) is that the applied forces be such that $L(\boldsymbol{\eta}) = 0$ for all the vector fields $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega)$ that satisfy

$$\frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) = 0 \quad \text{and} \quad \partial_{\alpha\beta}\eta_3 = 0 \quad \text{in } \omega. \tag{2.2}$$

It is therefore natural that we likewise begin by identifying *necessary conditions* for the existence of a minimizer of the functional J of (1.2) over the space $\boldsymbol{V}(\omega)$ defined in (1.1) (like in the linear case, these conditions will eventually turn out to be also sufficient when the norms $\|p_\alpha\|_{L^2(\omega)}$ are small enough; cf. Theorem 4.1).

In what follows, $\mathbb{M}^2, \mathbb{S}^2, \mathbb{S}^2_\geqslant$, and $\mathbb{S}^2_>$ respectively designate the set of all 2×2 real matrices, and of all symmetric, non-negative definite symmetric, and positive-definite symmetric, 2×2 real matrices.

Theorem 2.1. *In order that*

$$\inf \boldsymbol{\eta} \in \boldsymbol{V}(\omega) J(\boldsymbol{\eta}) > -\infty, \tag{2.3}$$

it is necessary that the vector fields

$$\boldsymbol{p} = (\boldsymbol{p}_H, p_3) := (p_i) \in \boldsymbol{L}^2(\omega) \quad \textit{and} \quad \boldsymbol{q}_H := (q_\alpha) \in \boldsymbol{L}^2(\omega)$$

satisfy the following relations. First,

$$\int_\omega \boldsymbol{p}(y)\,\mathrm{d}\omega = \boldsymbol{0}. \tag{2.4}$$

Second, define the matrix

$$\boldsymbol{A}(\boldsymbol{p}_H) := \int_\omega \boldsymbol{p}_H(y)\boldsymbol{y}^T \, \mathrm{d}\omega \in \mathbb{M}^2. \tag{2.5}$$

Then one of the following three mutually exclusive conditions is satisfied. If $\boldsymbol{A}(\boldsymbol{p}_H) = \boldsymbol{0}$, then

$$\boldsymbol{p}_H = \boldsymbol{0} \text{ a.e. in } \omega \quad \text{and} \quad \int_\omega \big(p_3\boldsymbol{y} - \boldsymbol{q}_H(y)\big)\, \mathrm{d}\omega = \boldsymbol{0}. \tag{2.6}$$

If $\mathbf{Ker}\, \boldsymbol{A}(\boldsymbol{p}_H) \neq \{\boldsymbol{0}\}$, then

$$\boldsymbol{A}(\boldsymbol{p}_H) \in \mathbb{S}^2_{\geqslant}, \boldsymbol{p}_H \in \big(\mathbf{Ker}\, \boldsymbol{A}(\boldsymbol{p}_H)\big)^\perp \text{ a.e. in } \omega \quad \text{and} \quad \int_\omega \big(p_3\boldsymbol{y} - \boldsymbol{q}_H(y)\big)\, \mathrm{d}\omega \in \big(\mathbf{Ker}\, \boldsymbol{A}(\boldsymbol{p}_H)\big)^\perp \text{ a.e. in } \omega. \tag{2.7}$$

If $\boldsymbol{A}(\boldsymbol{p}_H) \neq 0$ and $\mathbf{Ker}\, \boldsymbol{A}(\boldsymbol{p}_H) = \{\boldsymbol{0}\}$, then

$$\boldsymbol{A}(\boldsymbol{p}_H) \in \mathbb{S}^2_{>}. \tag{2.8}$$

Sketch of proof. In [3, Theorem 2.1], we showed that, if two vector fields $\tilde{\boldsymbol{\eta}} = (\tilde{\boldsymbol{\eta}}_H, \tilde{\eta}_3) \in \boldsymbol{V}(\omega)$ and $\boldsymbol{\eta} = (\boldsymbol{\eta}_H, \eta_3) \in \boldsymbol{V}(\omega)$ satisfy

$$\frac{1}{2}(\partial_\alpha \tilde{\eta}_\beta + \partial_\beta \tilde{\eta}_\alpha + \partial_\alpha \tilde{\eta}_3 \partial_\beta \tilde{\eta}_3) = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) \quad \text{and} \quad \partial_{\alpha\beta}\tilde{\eta}_3 = \partial_{\alpha\beta}\eta_3 \quad \text{in } L^2(\omega),$$

then

$$\tilde{\boldsymbol{\eta}}(y) = \boldsymbol{\eta}(y) + \boldsymbol{a} + b\boldsymbol{e} \wedge \boldsymbol{y} - \eta_3(y)\boldsymbol{d} + (\boldsymbol{d}\cdot\boldsymbol{y})\boldsymbol{e} - \frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{y})\boldsymbol{d} \quad \text{for almost all } y \in \omega,$$

for some $\boldsymbol{a} \in \mathbb{R}^3$, $b \in \mathbb{R}$, and $\boldsymbol{d} \in \mathbb{R}^2$, where $(\boldsymbol{e})_i := \delta_{i3}$. The proof thus amounts to finding necessary and sufficient conditions guaranteeing that the following two conditions simultaneously hold. *First*,

$$\sup\big\{L(\boldsymbol{a} + b\boldsymbol{e} \wedge \boldsymbol{id}); \boldsymbol{a} \in \mathbb{R}^3, b \in \mathbb{R}\big\} < +\infty. \tag{2.9}$$

Second,

$$\sup\big\{L\big(\boldsymbol{r}(\boldsymbol{d}, \eta_3)\big); \boldsymbol{d} \in \mathbb{R}^2\big\} < +\infty \quad \text{for each } \boldsymbol{\eta} \in \boldsymbol{V}(\omega), \tag{2.10}$$

where

$$\boldsymbol{r}(\boldsymbol{d}, \eta_3) := -\eta_3\boldsymbol{d} + (\boldsymbol{d}\cdot\boldsymbol{id})\boldsymbol{e} - \frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{id})\boldsymbol{d} \in \mathbb{R}^3.$$

Since $\{\boldsymbol{a} + b\boldsymbol{e} \wedge \boldsymbol{id}; \boldsymbol{a} \in \mathbb{R}^3, b \in \mathbb{R}\}$ is a vector space, condition (2.9) is equivalent to

$$L(\boldsymbol{a} + b\boldsymbol{e} \wedge \boldsymbol{id}) = 0 \quad \text{for all } \boldsymbol{a} \in \mathbb{R}^3 \text{ and } b \in \mathbb{R}.$$

Since

$$L(\boldsymbol{a} + b\boldsymbol{e} \wedge \boldsymbol{id}) = \int_\omega \boldsymbol{p}(y)\cdot\boldsymbol{a}\, \mathrm{d}\omega + b\int_\omega (-p_1 y_2 + p_2 y_1)\, \mathrm{d}\omega,$$

it follows that (2.9) is satisfied if and only if

$$\int_\omega \boldsymbol{p}(y)\, \mathrm{d}\omega = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{p}_H) := \int_\omega \boldsymbol{p}_H(y)\boldsymbol{y}^T \, \mathrm{d}\omega \in \mathbb{S}^2. \tag{2.11}$$

It is easily verified that, for each $\boldsymbol{d} \in \mathbb{R}^2$ and each $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$,

$$L\big(\boldsymbol{r}(\boldsymbol{d}, \eta_3)\big) = \boldsymbol{d}\cdot\left(\boldsymbol{s}(\boldsymbol{p}, \boldsymbol{q}_H, \eta_3) - \frac{1}{2}\boldsymbol{A}(\boldsymbol{p}_H)\boldsymbol{d}\right), \tag{2.12}$$

where

$$\boldsymbol{s}(\boldsymbol{p}, \boldsymbol{q}_H, \eta_3) := \int_\omega \big(p_3(y)\boldsymbol{y} - \boldsymbol{q}_H(y) - \eta_3(y)\boldsymbol{p}_H(y)\big)\, \mathrm{d}\omega \in \mathbb{R}^2. \tag{2.13}$$

If $\boldsymbol{A}(\boldsymbol{p}_H) \neq \boldsymbol{0}$, assume that there exists a vector $\boldsymbol{\delta}^\perp \in (\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp$ such that $\boldsymbol{\delta}^\perp \cdot \boldsymbol{A}(\boldsymbol{p}_H)\boldsymbol{\delta}^\perp < 0$. Then relations (2.10) cannot hold since

$$\sup_{t\in\mathbb{R}}\left\{t\boldsymbol{\delta}^\perp \cdot \boldsymbol{s}(\boldsymbol{p}, \boldsymbol{q}_H, \eta_3) - \frac{1}{2}t^2\boldsymbol{\delta}^\perp \cdot \boldsymbol{A}(\boldsymbol{p}_H)\boldsymbol{\delta}^\perp\right\} = +\infty.$$

Therefore, *the symmetric matrix $\boldsymbol{A}(\boldsymbol{p}_H)$ is necessarily either positive-definite if it is invertible, or non-negative-definite if it is singular*.

If $\boldsymbol{A}(\boldsymbol{p}_H)$ is singular (in which case $\boldsymbol{A}(\boldsymbol{p}_H) \in \mathbb{S}^2_\geqslant$), let $\boldsymbol{\delta} \in \mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H)$ be such that $\boldsymbol{\delta} \neq \boldsymbol{0}$. Expressing that (2.10) must hold in particular for any vector $\boldsymbol{d}$ of the form $\boldsymbol{d} = t\boldsymbol{\delta}$, $t \in \mathbb{R}$, then shows that, for each $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$, the vector $\boldsymbol{s}(\boldsymbol{p}, \boldsymbol{q}_H, \eta_3)$ must be orthogonal to $\boldsymbol{\delta}$. In other words, *if $\boldsymbol{A}(\boldsymbol{p}_H)$ is singular, then*

$$\boldsymbol{s}(\boldsymbol{p}, \boldsymbol{q}_H, \eta_3) \in \big(\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H)\big)^\perp \quad \text{for each } \boldsymbol{\eta} \in \boldsymbol{V}(\omega). \tag{2.14}$$

We now show that, *conversely*, if either $\boldsymbol{A}(\boldsymbol{p}_H) \in \mathbb{S}^2_>$, or $\boldsymbol{A}(\boldsymbol{p}_H) \in \mathbb{S}^2_\geqslant$ is singular and relation (2.14) holds, then relation (2.10) holds. First, we note that, if $\boldsymbol{A}(\boldsymbol{p}_H) = \boldsymbol{0}$, then $\boldsymbol{s}(\boldsymbol{p}, \boldsymbol{q}_H, \eta_3) = 0$ for each $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ by (2.12); hence $L(\boldsymbol{r}(\boldsymbol{d}, \eta_3)) = 0$ for each $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ and thus (2.10) holds in this case. Second, assume that $\boldsymbol{A}(\boldsymbol{p}_H) \neq \boldsymbol{0}$. Given any vector $\boldsymbol{d} \in \mathbb{R}^2$, let $\boldsymbol{d} = \boldsymbol{\delta} + \boldsymbol{\delta}^\perp$ with $\boldsymbol{\delta} \in \mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H)$ and $\boldsymbol{\delta}^\perp \in (\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp$. Then

$$L\big(\boldsymbol{r}(\boldsymbol{d}, \eta_3)\big) = \boldsymbol{\delta}^\perp \cdot \boldsymbol{s}(\boldsymbol{p}, \boldsymbol{q}_H, \eta_3) - \frac{1}{2}\boldsymbol{\delta}^\perp \cdot \big(\boldsymbol{A}(\boldsymbol{p}_H)\boldsymbol{\delta}^\perp\big) \leqslant \big|\boldsymbol{s}(\boldsymbol{p}, \boldsymbol{q}_H, \eta_3)\big|\big|\boldsymbol{\delta}^\perp\big| - \frac{\lambda}{2}\big|\boldsymbol{\delta}^\perp\big|^2,$$

where $|\cdot|$ denotes the Euclidean norm and $\lambda > 0$ denotes the smallest nonzero eigenvalue of the matrix $\boldsymbol{A}(\boldsymbol{p}_H)$. Hence $\sup_{\boldsymbol{d}\in\mathbb{R}^2} L(\boldsymbol{r}(\boldsymbol{d}, \eta_3)) < +\infty$ for each $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$, i.e., (2.10) also holds in this case.

The specific form of the vector $\boldsymbol{s}(\boldsymbol{p}, \boldsymbol{q}_H, \eta_3)$ (cf. (2.13)) then implies that relations (2.14) hold for all $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ if and only if $\int_\omega (p_3\boldsymbol{y} - \boldsymbol{q}_H(y))\,\mathrm{d}\omega \in (\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp$ *and* $\boldsymbol{p}_H \in (\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp$ a.e. in ω (hence $\boldsymbol{p}_H = \boldsymbol{0}$ a.e. in ω if $\boldsymbol{A}(\boldsymbol{p}_H) = \boldsymbol{0}$). This completes the proof. □

3. The intrinsic approach to the Neumann problem for a nonlinearly elastic plate

We now recast the minimization problem $\inf_{\boldsymbol{\eta}\in\boldsymbol{V}(\omega)} J(\boldsymbol{\eta})$, where the space $\boldsymbol{V}(\omega)$ and the functional $J : \boldsymbol{V}(\omega) \to \mathbb{R}$ are defined in (1.1)–(1.2), as a minimization problem in terms of the new unknowns $E_{\alpha\beta} \in L^2(\omega)$ and $F_{\alpha\beta} \in L^2(\omega)$ defined in (1.5). Crucial to this objective is the following result from [3]:

Theorem 3.1. *Let ω be a simply-connected domain in $\mathbb{R}^2$. Define the space*

$$\mathbb{E}(\omega) := \big\{(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega);\ \partial_{\sigma\tau}E_{\alpha\beta} + \partial_{\alpha\beta}E_{\sigma\tau} - \partial_{\alpha\sigma}E_{\beta\tau} - \partial_{\beta\tau}E_{\alpha\sigma} = F_{\alpha\sigma}F_{\beta\tau} - F_{\alpha\beta}F_{\sigma\tau} \text{ in } H^{-2}(\omega) \\ \text{and } \partial_\sigma F_{\alpha\beta} = \partial_\beta F_{\alpha\sigma} \text{ in } H^{-1}(\omega)\big\}. \tag{3.1}$$

Then, given any $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{E}(\omega)$, there exists a unique vector field

$$\boldsymbol{\eta} \in \boldsymbol{V}^0(\omega) := \left\{\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega);\ \int_\omega \boldsymbol{\eta}\,\mathrm{d}\omega = \boldsymbol{0}, \int_\omega \partial_\alpha\eta_3\,\mathrm{d}\omega = 0, \int_\omega (\partial_1\eta_2 - \partial_2\eta_1)\,\mathrm{d}\omega = 0\right\} \tag{3.2}$$

that satisfies

$$\frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3) = E_{\alpha\beta} \quad \text{in } L^2(\omega), \tag{3.3}$$

$$\partial_{\alpha\beta}\eta_3 = F_{\alpha\beta} \quad \text{in } L^2(\omega). \tag{3.4}$$

Some care must be exercised in applying this result however: Given a vector field $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$, the number $J(\boldsymbol{\eta})$ is *not* defined by the two fields $\boldsymbol{E} = (E_{\alpha\beta})$ and $\boldsymbol{F} = (F_{\alpha\beta})$ defined in (1.5), *because of the linear form L of* (1.3). As shown in the next theorem (whose proof relies on simple computations), the remedy consists in introducing a vector $\boldsymbol{\delta}^\perp$ in the subspace $(\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp$ of $\mathbb{R}^2$ as an *additional variable* (the matrix $\boldsymbol{A}(\boldsymbol{p}_H) \in \mathbb{S}^2_\geqslant$ is defined in (2.5)).

Theorem 3.2. *Given any $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$, let $\boldsymbol{\eta}^0 \in \boldsymbol{V}^0(\omega)$ be the unique vector field that satisfies* (*Theorem* 3.1)

$$\frac{1}{2}\big(\partial_\alpha\eta^0_\beta + \partial_\beta\eta^0_\alpha + \partial_\alpha\eta^0_3\partial_\beta\eta^0_3\big) = \frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3) \quad \text{and} \quad \partial_{\alpha\beta}\eta^0_3 = \partial_{\alpha\beta}\eta_3 \quad \text{in } L^2(\omega), \tag{3.5}$$

so that (*cf.* [3, Theorem 2.1]),

$$\boldsymbol{\eta} = \boldsymbol{\eta}_0 + \boldsymbol{a} + b\boldsymbol{e} \wedge \boldsymbol{id} - \eta_3\boldsymbol{d} + (\boldsymbol{d}\cdot\boldsymbol{id})\boldsymbol{e} - \frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{id})\boldsymbol{d} \tag{3.6}$$

for some $\boldsymbol{a} \in \mathbb{R}^3$, $b \in \mathbb{R}$, and $\boldsymbol{d} \in \mathbb{R}^2$. For each $\boldsymbol{d} \in \mathbb{R}^2$ and each $\eta_3 \in H^2(\omega)$, let

$$\boldsymbol{r}(\boldsymbol{d}, \eta_3) := -\eta_3 \boldsymbol{d} + (\boldsymbol{d} \cdot \boldsymbol{id})\boldsymbol{e} - \frac{1}{2}(\boldsymbol{d} \cdot \boldsymbol{id})\boldsymbol{d}. \tag{3.7}$$

Then

$$J(\boldsymbol{\eta}) = J\big(\boldsymbol{\eta}^0\big) - L\big(\boldsymbol{r}\big(\boldsymbol{\delta}^\perp, \eta_3^0\big)\big), \tag{3.8}$$

where, for each $\boldsymbol{d} \in \mathbb{R}^2$, the vector $\boldsymbol{\delta}^\perp$ denotes the projection of $\boldsymbol{d}$ onto the subspace $(\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp$ of $\mathbb{R}^2$.

Let the functional $\mathcal{J} : \mathbb{E}(\omega) \times (\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp \to \mathbb{R}$ be defined for each $((\boldsymbol{E}, \boldsymbol{F}), \boldsymbol{\delta}^\perp) \in \mathbb{E}(\omega) \times (\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp$ by

$$\mathcal{J}\big((\boldsymbol{E}, \boldsymbol{F}), \boldsymbol{\delta}^\perp\big) := \frac{1}{2} \int_\omega \left\{ \varepsilon a_{\alpha\beta\sigma\tau} E_{\sigma\tau} E_{\alpha\beta} + \frac{\varepsilon^3}{3} a_{\alpha\beta\sigma\tau} F_{\sigma\tau} F_{\alpha\beta} \right\} \mathrm{d}\omega - L\big(\boldsymbol{\Phi}(\boldsymbol{E}, \boldsymbol{F})\big) - L\big(\boldsymbol{r}\big(\boldsymbol{\delta}^\perp, \Phi_3(\boldsymbol{E}, \boldsymbol{F})\big)\big), \tag{3.9}$$

where $\boldsymbol{\Phi} = (\Phi_i) : \mathbb{E}(\omega) \to \boldsymbol{V}^0(\omega)$ is the nonlinear bijection defined for each $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{E}(\omega)$ by $\boldsymbol{\Phi}(\boldsymbol{E}, \boldsymbol{F}) := \boldsymbol{\eta}^0$, where $\boldsymbol{\eta}^0$ is the unique element in the space $\boldsymbol{V}^0(\omega)$ that satisfies Eqs. (3.3)–(3.4) (Theorem 3.1). Then the *intrinsic approach to the Neumann problem for a nonlinearly elastic plate consists in minimizing the function $\mathcal{J}$ of* (3.9) *over the set* $\mathbb{E}(\omega) \times (\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp$.

Note that, expressed in terms of $\boldsymbol{\eta}^0 = \boldsymbol{\Phi}(\boldsymbol{E}, \boldsymbol{F})$, the last two terms in (3.9) respectively become:

$$L\big(\boldsymbol{\Phi}(\boldsymbol{E}, \boldsymbol{F})\big) = \int_\omega \boldsymbol{p} \cdot \boldsymbol{\eta}^0 \,\mathrm{d}\omega - \int_\omega \boldsymbol{q}_H \cdot \nabla \eta_3^0 \,\mathrm{d}\omega, \tag{3.10}$$

$$L\big(\boldsymbol{r}\big(\boldsymbol{\delta}^\perp, \Phi_3(\boldsymbol{E}, \boldsymbol{F})\big)\big) = \boldsymbol{\delta}^\perp \cdot \int_\omega \big(p_3 \boldsymbol{y} - \boldsymbol{q}_H - \eta_3^0 \boldsymbol{p}_H\big) \,\mathrm{d}\omega - \frac{1}{2} \boldsymbol{\delta}^\perp \cdot \big(\boldsymbol{A}(\boldsymbol{p}_H)\boldsymbol{\delta}^\perp\big). \tag{3.11}$$

Note that *the additional variable $\boldsymbol{\delta}^\perp$ vanishes when $\boldsymbol{A}(\boldsymbol{p}_H) = \boldsymbol{0}$*, in which case the functional $\mathcal{J}$ of (3.9) reduces to its first two terms. But, even when $\boldsymbol{A}(\boldsymbol{p}_H) \neq \boldsymbol{0}$, it is still possible to define a minimization problem solely in terms of the variable $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{E}(\omega)$. To this end, notice that, for a *given* $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{E}(\omega)$, or equivalently for a *given* $\boldsymbol{\eta}^0 \in \boldsymbol{V}^0(\omega)$, there exists a unique $(\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp$ that minimizes the functional defined by

$$\boldsymbol{\delta}^\perp \in \big(\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H)\big)^\perp \subset \mathbb{R}^2 \quad \to \quad \mathcal{J}\big((\boldsymbol{F}, \boldsymbol{F}), \boldsymbol{\delta}^\perp\big) \in \mathbb{R},$$

since this quadratic functional is positive-definite in view of (3.11). Hence it is equivalent to minimize the functional $\mathcal{J} : \mathbb{E}(\omega) \to \mathbb{R}$ defined for each $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{E}(\omega)$ by

$$\tilde{\mathcal{J}}(\boldsymbol{E}, \boldsymbol{F}) := \inf_{\boldsymbol{\delta}^\perp \in (\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp} \mathcal{J}\big((\boldsymbol{E}, \boldsymbol{F}), \boldsymbol{\delta}^\perp\big).$$

4. Existence theorem

The next result constitutes the main result of this Note.

Theorem 4.1. *Assume that the vector fields $\boldsymbol{p} \in \boldsymbol{L}^2(\omega)$ and $\boldsymbol{q}_H \in \boldsymbol{L}^2(\omega)$ satisfy condition* (2.4) *and one of the conditions* (2.6), (2.7), *or* (2.8)*. Let the functional $\mathcal{J} : \mathbb{E}(\omega) \times (\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp \to \mathbb{R}$ be defined as in* (3.9)*. Then, if the norm $\|\boldsymbol{p}_H\|_{\boldsymbol{L}^2(\omega)}$ is small enough, there exists $((\bar{\boldsymbol{E}}, \bar{\boldsymbol{F}}), \bar{\boldsymbol{\delta}}^\perp) \in \mathbb{E}(\omega) \times (\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp$ such that*

$$\mathcal{J}\big((\bar{\boldsymbol{E}}, \bar{\boldsymbol{F}}), \bar{\boldsymbol{\delta}}^\perp\big) = \inf\big\{\mathcal{J}\big((\boldsymbol{E}, \boldsymbol{F}), \boldsymbol{\delta}^\perp\big); \big((\boldsymbol{E}, \boldsymbol{F}), \boldsymbol{\delta}^\perp\big) \in \mathbb{E}(\omega) \times \big(\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H)\big)^\perp\big\}. \tag{4.1}$$

Sketch of proof. (i) In [3, Theorem 3.1], it was shown that $\mathbb{E}(\omega)$ is sequentially weakly closed in $\mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$; hence *the set $\mathbb{E}(\omega) \times (\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp$ is sequentially weakly closed in $\mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega) \times \mathbb{R}^2$.*

(ii) *The functional $\mathcal{J}$ is sequentially weakly lower semi-continuous on the set $\mathbb{E}(\omega) \times (\mathbf{Ker}\,\boldsymbol{A}(\boldsymbol{p}_H))^\perp$.*

This property clearly holds for the functional defined by the first term in (3.9). That the mapping $\boldsymbol{\Phi}$ maps weakly convergent sequences in $\mathbb{E}(\omega)$ into strongly convergent sequences in $\boldsymbol{V}^0(\omega)$ (cf. Theorem 3.1 in [3]; in fact, weak convergence would suffice here) then implies that the functionals defined by the last two terms in (3.9) (re-expressed for this purpose as in (3.10)–(3.11)) are also sequentially weakly lower semi-continuous (recall that the variable $\boldsymbol{\delta}^\perp$ varies in a subspace of $\mathbb{R}^2$).

(iii) *The functional is coercive on* $\mathbb{E}(\omega) \times (\mathbf{Ker}\, \boldsymbol{A}(\boldsymbol{p}_H))^{\perp}$ *if the norm* $\|\boldsymbol{p}_H\|_{\boldsymbol{L}^2(\omega)}$ *is small enough.*

In what follows, $C_1, \ldots, C_6$, denote various constants that are independent of the various functions or vector fields involved in a given inequality, and the same notation $\|\cdot\|$ denotes the norm in the spaces $L^2(\omega)$, $\boldsymbol{L}^2(\omega)$, and $\mathbb{L}^2(\omega)$. Let the vector field $\boldsymbol{f} \in \boldsymbol{L}^2(\omega)$ be defined by

$$\boldsymbol{f}(y) := p_3(y)\boldsymbol{y} - \boldsymbol{q}_H(y) \quad \text{for almost all } y \in \omega.$$

Then, for any $((\boldsymbol{E}, \boldsymbol{F}), \boldsymbol{\delta}^{\perp}) \in \mathbb{E}(\omega) \times (\mathbf{Ker}\, \boldsymbol{A}(\boldsymbol{p}_H))^{\perp}$,

$$\mathcal{J}\big((\boldsymbol{E}, \boldsymbol{F}), \boldsymbol{\delta}^{\perp}\big) \geqslant C_1\|\boldsymbol{E}\|^2 + C_2\|\boldsymbol{F}\|^2 + \frac{\lambda}{2}\big|\boldsymbol{\delta}^{\perp}\big|^2 - \|\boldsymbol{p}_H\|\big\|\boldsymbol{\eta}_H^0\big\| - \|p_3\|\big\|\eta_3^0\big\| \\ - \|\boldsymbol{q}_H\|\big\|\nabla\eta_3^0\big\| - \|\boldsymbol{f}\|_{\boldsymbol{L}^1(\omega)}\big|\boldsymbol{\delta}^{\perp}\big| - \|\boldsymbol{p}_H\|\big\|\eta_3^0\big\|\big|\boldsymbol{\delta}^{\perp}\big|,$$

where $\lambda > 0$ denotes the smallest nonzero eigenvalue of the matrix $\boldsymbol{A}(\boldsymbol{p}_H) \in \mathbb{S}^2_{\geqslant}$ (recall that $\boldsymbol{\delta}^{\perp} = \mathbf{0}$ if $\boldsymbol{A}(\boldsymbol{p}_H) = \mathbf{0}$). Thanks to the relations that led to the *nonlinear Korn inequality* established in [3] (cf. Eqs. (3.2)–(3.5) in [3]),

$$\big\|\eta_3^0\big\|_{H^2(\omega)} \leqslant C_3\big\|\nabla^2\eta_3^0\big\| \quad \text{and} \quad \big\|\boldsymbol{\eta}_H^0\big\|_{\boldsymbol{H}^1(\omega)} \leqslant C_4\left(\left\|\nabla_s\boldsymbol{\eta}_H^0 + \frac{1}{2}\nabla\eta_3^0\nabla\eta_3^{0\,T}\right\| + \big\|\nabla^2\eta_3\big\|^2\right).$$

Therefore,

$$\mathcal{J}\big((\boldsymbol{E}, \boldsymbol{F}), \boldsymbol{\delta}^{\perp}\big) \geqslant C_1\|\boldsymbol{E}\|^2 + \big(C_2 - (C_5 + C_6/2)\|\boldsymbol{p}_H\|\big)\|\boldsymbol{F}\|^2 + \big(\lambda/2 - C_6/2\|\boldsymbol{p}_H\|\big)\big|\boldsymbol{\delta}^{\perp}\big|^2 \\ - C_5\|\boldsymbol{p}_H\|\|\boldsymbol{E}\| - C_6\big(\|p_3\| + \|\boldsymbol{q}_H\|\big)\|\boldsymbol{F}\| - \|\boldsymbol{f}\|_{\boldsymbol{L}^1(\omega)}\big|\boldsymbol{\delta}^{\perp}\big|.$$

Hence $\mathcal{J}$ is coercive on $\mathbb{E}(\omega) \times (\mathbf{Ker}\, \boldsymbol{A}(\boldsymbol{p}_H))^{\perp}$ if $\|\boldsymbol{p}_H\|$ is small enough. □

Note that, thanks to Theorem 3.2, the above theorem also establishes the *existence of a minimizer of the functional* J *of* (1.2) *in the space* $V(\omega)$ *of* (1.1), thus extending to the pure Neumann problem the existence result of [2] for the Dirichlet–Neumann (in which case the norm $\|\boldsymbol{p}_H\|_{L^2(\omega)}$ was also assumed to be small enough).

5. A definition of polyconvexity

Inspired by the definition proposed by John Ball in his landmark paper [1], we now propose a definition of *polyconvexity* that is directly adapted to the problem considered in this Note (this definition can be extended to more general situations; cf. [4]). For simplicity, we assume here that $\boldsymbol{A}(\boldsymbol{p}_H) = \mathbf{0}$.

Let the subset $\mathbb{E}(\omega)$ of $\mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$ be defined as in (3.1). Then an integrand $W : \mathbb{E}(\omega) \to \mathbb{R}$ is said to be *polyconvex* if there exists a *convex* function $\mathbb{W} : \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$ such that

$$W(\boldsymbol{E}, \boldsymbol{F}) = \mathbb{W}(\boldsymbol{E}, \boldsymbol{F}) \quad \text{for all } (\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{E}(\omega).$$

Clearly, this assumption is satisfied by the integrand appearing in the first term of the functional $\mathcal{J}$ of (3.9).

To further substantiate this definition, it will be proved in [4] that *the set* $\mathbb{E}(\omega)$ *is a manifold of class* $\mathcal{C}^{\infty}$ *in* $\mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$ and that *the mapping* $\boldsymbol{\Phi}^{-1} : \boldsymbol{V}^0(\omega) \to \mathbb{E}(\omega)$ *is a* $\mathcal{C}^{\infty}$*-diffeomorphism*, where the space $\boldsymbol{V}^0(\omega)$ and the mapping $\boldsymbol{\Phi}$ are defined as in Section 3.

The proof of Theorem 4.1 then shows that this notion is indeed the key to establishing the coerciveness and the sequential weak lower semi-continuity of the functional $\mathcal{J}$, just like the notion of polyconvexity introduced by John Ball in nonlinear three-dimensional elasticity.

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References

[1] J. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Arch. Ration. Mech. Anal. 63 (1977) 337–403.
[2] P.G. Ciarlet, P. Destuynder, A justification of a nonlinear model in plate theory, Comput. Methods Appl. Mech. Engrg. 17/18 (1979) 227–258.
[3] P.G. Ciarlet, S. Mardare, Nonlinear Saint-Venant compatibility conditions for nonlinearly elastic plates, C. R. Acad. Sci. Paris, Ser. I 349 (23–24) (2011) 1297–1302.
[4] P.G. Ciarlet, S. Mardare, Saint-Venant compatibility conditions and a notion of polyconvexity in nonlinear plate theory, in preparation.