Differential Topology/Calculus of Variations

Prescribing the Webster scalar curvature on CR spheres

Préscription de la courbure scalaire de Webster sur la sphère CR

Ridha Yacoub

Département de mathématique-informatique, I.P.E.I.M., rue Ibn Al Jazzar, 5019 Monastir, Tunisia

1. Introduction

In this Note, we revisit a problem of geometric origin. Namely, let \( S^{2n+1} \) be the unit sphere of \( \mathbb{C}^{n+1} \) defined by \( S^{2n+1} = \{ \zeta = (\zeta^1, \ldots, \zeta^{n+1}) \in \mathbb{C}^{n+1} / |\zeta|^2 = \sum_{j=1}^{n+1} |\zeta_j|^2 = 1 \} \) endowed with its standard contact form \( \theta = i(\partial - \bar{\partial})|\zeta|^2 = i \sum_{j=1}^{n+1} \zeta_j d\bar{\zeta}_j - \bar{\zeta}_j d\zeta_j \). The Webster scalar curvature problem resumes to find a contact form \( \tilde{\theta} \) CR equivalent to \( \theta \) such that the associated Webster scalar curvature \( R_{\tilde{\theta}} = K \), where \( K \) is some given function on \( S^{2n+1} \). It is equivalent to find a solution \( u \) of the following problem:

\[
\begin{cases}
Lu = Ku^{1+\frac{2}{n}}, \\
u > 0 \quad \text{on} \quad S^{2n+1}
\end{cases}
\]

where \( L = \frac{2(n+1)}{n} \Delta_0 + \frac{n(n+1)}{2} \) is the conformal sub-Laplacian (since the Webster scalar curvature of \( \theta \) is \( R_0 = \frac{n(n+1)}{4} \)), and \( \Delta_0 \) denoted also by \( \Delta_0 \) in the literature is the sub-Laplacian operator (the real part of the Kohn Laplacian \( \Box_0 \)) for \( (S^{2n+1}, \theta) \). Note that problem (1) is the CR counterpart of the scalar curvature problem in the Riemannian setting (see e.g. [2,8,10]).

This problem has been envisaged under perturbation or symmetric hypotheses (see [5] and [9]). Our aim is to handle such a question using some topological and dynamical tools related to the theory of critical points at infinity (see Bahri [1]).

Let \( K : S^{2n+1} \rightarrow \mathbb{R} \) be a \( C^2 \) function. We introduce the following assumption:

\( (N.D) \) \( K \) has a finite set of nondegenerate critical points, denoted by \( \mathcal{K} \), such that \( \Delta_0 K(y) \neq 0, \quad \forall y \in \mathcal{K} \).

E-mail address: ridha.yacoub@ipeim.rnu.tn.
We introduce the subset:
\[ K^+ = \{ y \in K; -\Delta_y K(y) > 0 \}, \]
and we denote by (C.C) the condition:

(C.C) Assume that \( K(\xi) = 1 + e K_0(\xi) \). \( \forall \xi \in \mathbb{S}^{2n+1} \), where \( K_0 \in C^2(\mathbb{S}^{2n+1}) \) and \( |\epsilon| \) small.

Let \( ind(K, y) \) be the Morse index of \( K \) at its critical point \( y \). For an integer \( k \), we write \( k \in \mathbb{N} \), if \( k \) satisfies the following condition: For any \( y \in K^+ \), we have \( 2n + 1 - ind(K, y) \neq k + 1 \). That is
\[ \mathbb{N} = \{ k \in \mathbb{N} / \forall y \in K^+, 2n + 1 - ind(K, y) \neq k + 1 \}. \]
For example if \( k \geq 2n + 1 \) then \( k \in \mathbb{N} \).

Our first existence result generalizes that of [9]:

**Theorem 1.1.** Let \( n \geq 1, K : \mathbb{S}^{2n+1} \rightarrow \mathbb{R} \) a \( C^2 \) positive function satisfying (N.D) and (C.C). If
\[ \max_{k \in \mathbb{N}} \left| 1 - \sum_{y \in K^+} (-1)^{2n+1-ind(K, y)} \right| \neq 0 \quad (4) \]
then, for \( |\epsilon| \) small enough, there exists a solution for problem (1).

As a corollary we find the result of [9]:

**Corollary 1.2.** Let \( n \geq 1, K : \mathbb{S}^{2n+1} \rightarrow \mathbb{R} \) a \( C^2 \) positive function satisfying (N.D) and (C.C). If
\[ \sum_{y \in K^+} (-1)^{2n+1-ind(K, y)} \neq 1, \quad (5) \]
then, for \( |\epsilon| \) small enough, there exists a solution for problem (1).

Our second existence result is not based on the count-indices formula. We introduce
\[ \kappa^+_p = \{ y \in K^+; ind(K, y) = p \}, \quad \text{and} \quad \kappa^+_\omega = \kappa^+ \setminus \kappa^+_0. \]

**Theorem 1.3.** Let \( n \geq 1, K : \mathbb{S}^{2n+1} \rightarrow \mathbb{R} \) a \( C^2 \) positive function satisfying (N.D), (C.C), and (A) \( \exists \tilde{y} \in \kappa^+_1 \), such that, \( K(\tilde{y}) \geq K(z), \forall z \in \kappa^+_2 \).

Then, provided that \( |\epsilon| \) is small enough, there exists a solution for problem (1).

2. Variational framework. Lack of compactness. Proofs of the results

Let \( S^1 \) be the completion of \( C^\infty(\mathbb{S}^{2n+1}) \) for the norm \( \| u \| = \int_{\mathbb{S}^{2n+1}} L u u \theta \wedge d\theta^n \). Let \( \sum = \{ u \in S^1 / \| u \| = 1 \} \) and \( \sum^+ = \{ u \in \sum / u \geq 0 \} \). The Euler functional associated to problem (1) on \( S^1 \) is:
\[ J(u) = \frac{\| u \|^2}{\int_{\mathbb{S}^{2n+1}} K(u)^{2+2^\frac{2}{n}} \theta \wedge d\theta^n} \frac{n}{n-\frac{2}{n}}. \]

One knows that if \( v \) is a critical point of \( J \) in \( \sum^+ \), then \( u = J(v)^\frac{1}{2} v \) is a solution for (1) in \( S^1 \), and hence the contact form \( \tilde{\theta} = u \theta \) has its Webster scalar curvature \( R_{\tilde{\theta}} = K \).

Problem (1) is known to be delicate because the inclusion \( S^1 \hookrightarrow L^{\frac{2n+4}{n-2}} \) is continuous but not compact, and the functional \( J \) does not satisfy the Palais–Smale condition. In order to characterize the sequences failing the Palais–Smale condition, we recall some definitions and notations. Let \( \omega \) be the solution of Yamabe problem on the Heisenberg group \( \mathbb{H}^n \), defined for all \( \xi = (z, t) \) in \( \mathbb{H}^n \) by \( \omega(\xi) = |1 + |z|^2 - it|^{-n} \). For each \( (g, \lambda) \in \mathbb{H}^n \times (0, \infty) \) we obtain the other solutions \( \omega_{(g, \lambda)}(\xi) = \lambda^n \omega(\lambda^{-\frac{1}{2}} \xi) \). Now, for each \( (a, \lambda) \in \mathbb{S}^{2n+1} \times (0, \infty) \), we introduce the solution of Yamabe problem on \( \mathbb{S}^{2n+1} \):
\[ \delta(a, \lambda)(\xi) = \frac{1}{|1 + \xi^{n+1} |^n} \omega(F(a, \lambda) \circ F(\xi)) \]
where \( F \) is a biholomorphic map from \( \mathbb{S}^{2n+1} \) onto \( \mathbb{H}^n \), induced by the Cayley Transform (see [7,9]). The sphere is globally CR equivalent to itself; then, standard arguments and results of [3] and [4] (where the case of manifolds locally CR-equivalent to the sphere was treated) are valid here. Since we are arguing by contradiction we have
Proposition 2.1. Assume (1) has no solution. The only critical points at infinity of $J$ in $\Sigma^+$ are combinations of $q$ masses ($q \geq 1$), $\sum_{i=1}^{q} \delta(y_i, \infty) := (y_1, \ldots, y_q, \infty)$, where the $y_i$’s are distinct in $\mathbb{K}^+$. The Morse index of $(y_1, \ldots, y_q, \infty)$ is $m(y_1, \ldots, y_q, \infty) = q - 1 + \sum_{i=1}^{q} 2n + 1 - \text{ind}(K, y_i)$.

Proof of Theorem 1.1. Since $K = 1 + \varepsilon K_0$, for $\varepsilon = 0$ we obtain the Yamabe functional $J_0$, which possesses a $(2n + 2)$-dimensional manifold of critical points $Z = \{K(y_1, a, \lambda) \in \mathbb{R}^{2n+1} \times (0, \infty)\}$. For $\beta \in \mathbb{R}$, we have $J_0 = \{u \in \Sigma, \ J(u) \leq \beta\}$. Observe that, since $K_0$ is bounded, $J(u) = J_0(u)(1 + O(\varepsilon))$, where $O(\varepsilon)$ is independent of $u$ and tends to $0$ with $\varepsilon$. Readily we derive:

Lemma 2.1. Let $\eta > 0$, for $|\varepsilon|$ small enough, we have $J^s + \eta \subset J^{s+2\eta} \subset J^{5+3\eta}$.

Proof of Theorem 1.3. Arguing by contradiction, we assume that (1) has no solution. Let $\partial$ be the boundary operator in the sense of Floer-Minlos homology as introduced in [11]. We recall that singular chains, in this homology, are generated by unstable manifolds of critical points of $J$, and, if $(y)_\infty$ is a critical point at infinity of Morse index $m(y)_\infty$, then

$$\partial(W_u(y)_\infty) = \sum_{i = 0}^{m(y)_\infty} i((y)_\infty, (z)_\infty) W_u(z)_\infty$$

where $i((y)_\infty, (z)_\infty)$ is the intersection number of $W_u(y)_\infty$ and $W_d(z)_\infty$, the unstable (resp. stable) manifold of $(y)_\infty$ (resp. $(z)_\infty$), with respect to $-\partial J$ [see [11]].

Taking $y = \tilde{y}$, defined in assumption (A) of Theorem 1.3, $W_u(\tilde{y})_\infty$ is a manifold of dimension $m(\tilde{y})_\infty = 2n$, and satisfies $W_u(\tilde{y})_\infty \cap W_d(z)_\infty = \emptyset$ for any $(z)_\infty$ of Morse index $2n - 1$, since under assumption (A), $J((y)_\infty) \leq J(z)_\infty$ for all $z \in \mathbb{K}_+^+$. It follows that $\partial W_u(\tilde{y})_\infty = 0$, and hence $W_u(\tilde{y})_\infty$ defines a cycle in $C_{2n}(X^-)$ the group of $2n$-dimensional chains of $X^- = \sum_{y \in \mathbb{K}_+^+} W_u(y)_\infty$. Note that $X^- = \sum_{y \in \mathbb{K}_+^+} W_u(y)_\infty$. That $X^- = \sum_{y \in \mathbb{K}_+^+} W_u(y)_\infty$. Therefore $W_u(\tilde{y})_\infty$ defines a homological class of dimension $2n$ which is nontrivial in $X^-$. Denoting by $H_{2n}(X^-)$ the $2n$-dimensional homology group of $X^-$, we then have

$$H_{2n}(X^-) \neq 0.$$  

Using the same arguments and notations of the proof of Theorem 1.1, we derive that $X^- = \text{contractible in } J^{5+\eta}$ which contradicts by deformation on $X^-$. It follows that $X^- = \text{contractible set, and therefore } H_k(X^-) = 0, \forall k \geq 1$, which is in contradiction with (8). This ends the proof of Theorem 1.3.
References