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Group Theory

Kirillov's formula and Guillemin–Sternberg conjecture

Formule de Kirillov et conjecture de Guillemin–Sternberg

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ABSTRACT

Let G be a connected reductive real Lie group, and H a compact connected subgroup. Let M be a coadjoint admissible orbit of G and let Π be one of the unitary irreducible representations of G attached to M by Harish-Chandra. Using the character formula for Π, we give a geometric formula for the multiplicities of the restriction of Π to H, when the restriction map $p : M \to \mathfrak{h}^*$ is proper. In particular, this gives an alternate proof of a result of Paradan.

RÉSUMÉ

Soit G un groupe de Lie réel réductif connexe, et H un sous-groupe compact connexe. Soit M une orbite coadjointe admissible de G et soit Π une des représentations unitaires irréductibles associées à M par Harish-Chandra. Grâce aux formules de caractère pour Π, nous donnons une formule géométrique pour les multiplicités de la restriction de Π à H lorsque l'application de restriction $p : M \to \mathfrak{h}^*$ est propre. En particulier, ceci donne une autre démonstration d'un résultat de Paradan.

1. Introduction

When M is a compact pre-quantizable Hamiltonian manifold for the action of a compact connected Lie group H with moment map $p : M \to \mathfrak{h}^*$, Guillemin and Sternberg defined a quantization of M, which is a virtual representation of H. They proposed formulae for the multiplicities in terms of the reduced "manifolds" $p^{-1}(\nu)/H(\nu)$. These formulae have been proved in [7], and in the close setting of quantization with ρ-correction (also called $Spin_c$-quantization) in [9]. In this Note we consider only quantization with ρ-correction.

When M is not compact, it is not clear how to define a quantization of M. In the case where M is a coadjoint admissible orbit, closed and of maximal dimension, of a real connected reductive Lie group G, the representations Π associated to M by Harish-Chandra are the natural candidates for the quantization of M. When H is the maximal compact subgroup of G, Paradan [8] has shown that the motto "quantization commutes with reduction" still holds for these non-compact Hamiltonian manifolds. We give another proof using character formulae. It holds for any connected compact subgroup H, provided the moment map $p : M \to \mathfrak{h}^*$ is proper. However, our proof uses a special feature of these manifolds M: their $\hat{A}$-genus is trivial. So it does not extend to representations associated to coadjoint orbits of G which are not of maximal dimension.

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2. Box splines and Dahmen–Micchelli deconvolution

Let V be a finite dimensional real vector space, $\Lambda \subset V$ a lattice, and $\mathrm{d}v$ the associated Lebesgue measure. For $v \in V$, we denote by δ_v the δ measure at v, by ∂_v the differentiation in the direction v. Let $\Phi = [\alpha_1, \alpha_2, \dots, \alpha_N]$ be a list of elements in Λ and let $\rho_\Phi = \frac{1}{2}\sum_{\alpha\in\Phi}\alpha$. The centered box spline $B_c(\Phi)$ is the measure on V such that, for a continuous function F on V,

$$\langle B_c(\Phi), F\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} F\left(\sum_{i=1}^{N} t_i\alpha_i\right) \mathrm{d}t_1 \cdots \mathrm{d}t_N.$$

The Fourier transform $\hat{B}_c(\Phi)(x)$ is the function $\prod_{\alpha\in\Phi} \frac{e^{i\langle\alpha,x\rangle/2} - e^{-i\langle\alpha,x\rangle/2}}{i\langle\alpha,x\rangle}$.

Define a series of differential operators on V by

$$\hat{A}(\Phi) = \prod_{\alpha\in\Phi} \frac{\partial_\alpha}{e^{\frac{1}{2}\partial_\alpha} - e^{-\frac{1}{2}\partial_\alpha}} = 1 - \frac{1}{24}\sum_\alpha (\partial_\alpha)^2 + \cdots.$$

We assume now that Φ generates V. A vector $\epsilon \in V$ is called generic if ϵ does not lie on any hyperplane U spanned by elements of Φ.

We choose some $\kappa \in V$. Usually, the choice of κ is clear from the context, and we do not show the dependence on κ. A point $v \in V$ is called Φ-regular if v does not lie on any affine hyperplane $\rho_\Phi + \kappa + \lambda + U$ where $\lambda \in \Lambda$. A connected component $\mathfrak{c}$ of the set V_{reg} of Φ-regular elements is called an alcove.

A piecewise polynomial function $\mathbf{h}$ on V is a function defined on the open set V_{reg} and such that, for any alcove $\mathfrak{c}$, there exists a polynomial function $h^{\mathfrak{c}}$ which coincide with $\mathbf{h}$ on $\mathfrak{c}$. If D is a differential operator (or a series of differential operators) with constant coefficients, we can define the piecewise differentiation $\mathbf{h} \mapsto D\mathbf{h}$, by applying D on each alcove to the function $\mathbf{h}$. Let ϵ generic. We define a function $\lim_\epsilon \mathbf{h}$ on V by $\lim_\epsilon \mathbf{h}(v) = h^{\mathfrak{c}}(v)$, where $\mathfrak{c}$ is the alcove such that $v + t\epsilon \in \mathfrak{c}$ for small $t > 0$. We can also translate a piecewise polynomial function by an element ξ of V (this will change κ to $\kappa + \xi$).

Consider the box spline $B_c(\Phi)$. For each alcove (here $\kappa = 0$) $\mathfrak{c}$, there exists a polynomial function $b^{\mathfrak{c}}$ on V such that the measure $B_c(\Phi)$ coincide with $b^{\mathfrak{c}}(v)\,\mathrm{d}v$ on $\mathfrak{c}$. Thus the collection of functions $b^{\mathfrak{c}}$ define a piecewise polynomial function $\mathbf{b}$.

Denote by $P = \kappa + \Lambda$ the translate of Λ, and $\mathcal{C}(P)$ the space of complex valued functions on P. If $m \in \mathcal{C}(P)$, the function

$$\mathbf{b}(m)(v) = \sum_{\lambda\in P} m(\lambda)\mathbf{b}(v - \lambda)$$

is a piecewise polynomial function such that

$$\mathbf{b}(m)\,\mathrm{d}v = \sum_{\lambda\in P} m(\lambda)\,\delta_\lambda * B_c(\Phi).$$

Recall that the list Φ is called unimodular if any basis of V contained in Φ is a basis of the lattice Λ.

Theorem 2.1. *(See [2].) Let $m \in \mathcal{C}(P)$ and let $\nu \in P$. If Φ is unimodular, then, for any ϵ generic, we have*

$$m(\nu) = \lim_\epsilon\big(\hat{A}(\Phi)\mathbf{b}(m)\big)(\nu).$$

However, we need to consider the general case, where Φ is not necessarily unimodular. We consider Λ as the group of characters of a torus T, and use the notation s^λ for the value of $\lambda \in \Lambda$ at $s \in T$. Let $\mathcal{V}(\Phi)$ be the set of $s \in T$ such that the list $\Phi_s = [\alpha, s^\alpha = 1]$ generates V (it is called the vertex set).

Consider a vertex $s \in \mathcal{V}(\Phi)$ and the convolution product

$$B_c(s, \Phi) = \left(\prod_{\alpha\in\Phi\setminus\Phi_s} \frac{\delta_{\alpha/2} - s^{-\alpha}\delta_{-\alpha/2}}{1 - s^{-\alpha}}\right) * B_c(\Phi_s). \tag{1}$$

If $m \in \mathcal{C}(P)$, Theorem 2.2 below (basically due to Dahmen–Micchelli) implies that we can recover the value of m at a point $\nu \in P$ from the knowledge, in a neighborhood of ν, of the locally polynomial measures (for all $s \in \mathcal{V}(\Phi)$)

$$\mathbf{b}(s, m, \kappa)\,\mathrm{d}v = \left(\sum_{\nu\in P} s^{\nu-\kappa} m(\nu)\delta_\nu\right) * B_c(s, \Phi). \tag{2}$$

Define the series of differential operators

$$E(s,\Phi)=\prod_{\alpha\in\Phi\setminus\Phi_s}\frac{1-s^{-\alpha}}{e^{\partial_{-\alpha/2}}-s^{-\alpha}e^{\partial_{\alpha/2}}}=1+\frac{1}{2}\sum_{\alpha\in\Phi\setminus\Phi_s}\frac{1+s^{-\alpha}}{1-s^{-\alpha}}\partial_\alpha+\cdots$$

and

$$\hat{A}(s,\Phi)=E(s,\Phi)\hat{A}(\Phi_s).$$

Theorem 2.2. *(See [3].) Let $m\in\mathcal{C}(P)$ and let $\nu\in P$. Then, for any ϵ generic, we have*

$$m(\nu)=\sum_{s\in\mathcal{V}(\Phi)}s^{\kappa-\nu}\lim_{\epsilon}\big(\hat{A}(s,\Phi)\mathbf{b}(s,m,\kappa)\big)(\nu).$$

3. Kirillov's formula

Let G be a connected reductive real Lie group with Lie algebra $\mathfrak{g}$. The function

$$j_{\mathfrak{g}}(X)=\det_{\mathfrak{g}}\left(\frac{e^{ad\,X/2}-e^{-ad\,X/2}}{ad\,X}\right)$$

admits a square root $j_{\mathfrak{g}}^{1/2}(X)$, an analytic function on $\mathfrak{g}$ with $j_{\mathfrak{g}}^{1/2}(0)=1$. Let s be a semi-simple element of G, and $\mathfrak{g}(s)$ its centralizer. The function $\det_{\mathfrak{g}/\mathfrak{g}(s)}(\frac{1-se^{ad\,X}}{1-s})$ admits a square root $D^{1/2}(s,X)$, an analytic function on $\mathfrak{g}(s)$ with $D^{1/2}(s,0)=1$.

Let H be a compact connected group, with Lie algebra $\mathfrak{h}$. Let T be a Cartan subgroup of H with Lie algebra $\mathfrak{t}$. We will apply the results of the previous paragraph to the vector space $V=\mathfrak{t}^*$ equipped with the lattice $\Lambda\subset\mathfrak{t}^*$ of weights of T (thus $e^{i\lambda}$ is a character of T). Let W be the Weyl group of (H,T). Choose a positive system $\Delta^+\subset\mathfrak{t}^*$ for the non-zero weights of the adjoint action of T in $\mathfrak{h}_{\mathbb{C}}$. For $X\in\mathfrak{t}$,

$$j_{\mathfrak{h}}^{1/2}(X)=\prod_{\alpha\in\Delta^+}\frac{e^{i\langle\alpha,X\rangle/2}-e^{-i\langle\alpha,X\rangle/2}}{i\langle\alpha,X\rangle}.$$

For κ, we will use $\rho_H=\rho_{\Delta^+}$. Let $\mathfrak{t}^*_+$ be the open Weyl chamber. Thus $\mathfrak{t}^*_+$ intersect every orbit of H in $\mathfrak{h}^*$ of maximal dimension in one point. Consider the set $P_{\mathfrak{h}}=(\rho_H+\Lambda)\subset\mathfrak{t}^*$ and $P_{\mathfrak{h}}^+=(\rho_H+\Lambda)\cap\mathfrak{t}^*_+$. A function *mult* on $P_{\mathfrak{h}}^+$ will be extended to a W-anti-invariant function m on $P_{\mathfrak{h}}$.

The set $P_{\mathfrak{h}}^+$ is in one-to-one correspondence $\mu\mapsto\Pi^H(\mu)$ with the dual $\hat{H}$ of H. The identity

$$\operatorname{Tr}\Pi^H(\mu)(\exp X)=\sum_{w\in W}\frac{\epsilon(w)e^{\langle iw\mu,X\rangle}}{\prod_{\alpha\in\Delta^+}e^{i\langle\alpha,X\rangle/2}-e^{-i\langle\alpha,X\rangle/2}}$$

holds on $\mathfrak{t}$. This is the Atiyah–Bott fixed-point formula for the index of a twisted Dirac operator on $H\mu$, so that $\Pi^H(\mu)$ is the quantization $Q(H\mu)$ of the symplectic manifold $H\mu$.

Let $p:M\to\mathfrak{h}^*$ be the moment map of a connected H-Hamiltonian manifold M. Let β_M be the Liouville measure. The *slice* S of M is the locally closed subset $p^{-1}(\mathfrak{t}^*_+)$ of M. It is a symplectic submanifold of M with associated Liouville measure β_S. If p is proper, the restriction p^0 of p to S defines a proper map $S\to\mathfrak{t}^*_+$. We extend the push-forward measure $p^0_*(\beta_S)$ on $\mathfrak{t}^*_+$ to a W-anti-invariant signed measure on $\mathfrak{t}^*$ denoted by $DH(M,p)$ (the Duistermaat–Heckman measure). If S is non-empty (that is, if $p(M)$ contains an H-orbit of maximal dimension), the support of $DH(M,p)$ is equal to $p(M)\cap\mathfrak{t}^*$. Suppose moreover that there exist regular values $\nu\in\mathfrak{t}^*_+$ of p^0. At such ν, the reduced space $M_\nu=p^{-1}(\nu)/H(\nu)$ is an orbifold with symplectic form denoted by Ω_ν, and corresponding Liouville measure β_{M_ν}. By [6], the measure $DH(M,p)$ has a polynomial density with respect to $d\nu$ in a neighborhood of ν, and the value at ν is the symplectic volume $\int_{M_\nu}e^{\Omega_\nu/2\pi}=\int_{M_\nu}\beta_{M_\nu}$.

Some unitary irreducible representations of G can similarly be associated to closed admissible orbits of maximal dimension of the coadjoint representation of G. Recall Harish-Chandra parametrization. To simplify, we assume G linear. Let $f_0\in\mathfrak{g}^*$ such that $\mathfrak{g}(f_0)$ (its centralizer in $\mathfrak{g}$) is a Cartan subalgebra of $\mathfrak{g}$. Denote by $\tilde{G}(f_0)$ the metaplectic two fold cover of the stabilizer $G(f_0)$ of f_0. Let τ be a character of $\tilde{G}(f_0)$ such that $\tau(\exp X)=e^{i\langle f,X\rangle}$ and $\tau(\epsilon)=-1$ if $\epsilon\in\tilde{G}(f_0)$ projects on 1 and $\epsilon\neq 1$ (if such a character τ exists, f_0 is called admissible). As explained in [4], it follows from deep work of many mathematicians, especially Harish-Chandra, that to this data is associated an irreducible unitary representation $\Pi^G(f_0,\tau)$ of G. We consider it as a quantization $Q(M,\tau)$ of M. If f_0 is admissible and $G(f_0)$ is connected (as is the case when $G(f_0)$ is compact), the character τ is unique, and we simply write $Q(M)$ for $Q(M,\tau)$.

Irreducible unitary representations of G have a character, which, by Harish-Chandra theory, is a locally L^1 function on G. We denote by $\Theta(M,\tau)$ the character of $Q(M,\tau)$. Similarly, the measure β_M, considered as a tempered measure on $\mathfrak{g}^*$, has a Fourier transform which is a locally L^1 function on $\mathfrak{g}$. Kirillov's formula (proven in this case by Rossmann [10]) is the equality of locally L^1 functions on $\mathfrak{g}$:

$$j_{\mathfrak{g}}^{1/2}(X)\,\Theta(M,\tau)(\exp X)=\int_M e^{i\langle f,X\rangle}\,\mathrm{d}\beta_M(f).$$

We suppose that the connected compact group H is a subgroup of G, and we assume that the projection map $p: M \to \mathfrak{h}^*$ is proper. It implies that the restriction

$$Q(M,\tau)|_H=\sum_{\mu\in P_{\mathfrak{h}}^+} mult(\mu)\Pi^H(\mu)$$

is a sum of irreducible representations of H with finite multiplicities $mult(\mu)$. We associated to $mult(\mu)$ an anti-invariant function $m(\mu)$ on $P_{\mathfrak{h}}$, and to the projection p an anti-invariant measure $DH(M,p)$ on $\mathfrak{t}^*$. Let $\Delta(\mathfrak{g}/\mathfrak{h})\subset\mathfrak{t}^*$ be the list of weights for the action of T in $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$. Choose a sublist Φ so that $\Delta(\mathfrak{g}/\mathfrak{h})$ is the disjoint union of Φ, $-\Phi$ and the zero weights. The subsequent definitions do not depend of this choice. On $\mathfrak{t}$, we have

$$j_{\mathfrak{g}}^{1/2}(X)=j_{\mathfrak{h}}^{1/2}(X)\prod_{\alpha\in\Phi}\frac{e^{i\langle\alpha,X\rangle/2}-e^{-i\langle\alpha,X\rangle/2}}{i\langle\alpha,X\rangle}.$$

We assume (and we can easily restrict to this case) that $\mathfrak{h}$ does not contain any ideal of $\mathfrak{g}$. Then the set Φ generates $\mathfrak{t}^*$. Kirillov's formula, written for the characters of $Q(M,\tau)$ and $Q(H\mu)$, implies the equality of measures on $\mathfrak{t}^*$:

$$\left(\sum_{\nu\in P_{\mathfrak{h}}} m(\nu)\delta_\nu\right)*B_c(\Phi)=DH(M,p).$$

We write $DH(M,p)=\mathbf{d}\,\mathrm{d}\nu$, where $\mathbf{d}$ is piecewise polynomial. When $\nu\in\mathfrak{t}^*$ is regular, we have $r(\nu)=(\hat{A}(\Phi)\mathbf{d})(\nu)=\int_{M_\nu} e^{\Omega_\nu/2\pi}\hat{A}(M_\nu)$ where $\hat{A}(M_\nu)$ is the $\hat{A}$-genus of M_ν. This follows from expressing the linear variation of Ω_ν in function of the curvature of the principal bundle $(p^0)^{-1}(\nu)/T$ [6].

In the (rare) case where the system Φ is unimodular (for example for G the adjoint group of $U(p,q)$, and H the maximal compact subgroup), the orbifold M_ν is smooth. The value $r(\nu)$ can be defined at any $\nu\in P_{\mathfrak{h}}$, by taking a limit of $r(\nu+t\epsilon)$ for any ϵ generic, and coincide with the number $Q(M_\nu)\in\mathbb{Z}$ defined as the quantization of the (possibly singular) reduction M_ν [8]. Thus we obtain

$$Q(M)|_H=\sum_{\nu\in P_{\mathfrak{h}}^+\cap p(M)} Q(M_\nu)Q(H\nu).$$

We now consider the general case. Consider a vertex $s\in T$ for Φ. Let $M(s)$ be the submanifold of M fixed by s. It may have several connected components. It is a symplectic submanifold, and we denote by β_s its Liouville measure. We can define the generalized function $\Theta(f,\tau)(sg)$ where $g\in G$ commutes with s. The identity

$$j_{\mathfrak{g}(s)}^{1/2}(X)D^{1/2}(s,X)\Theta(f,\tau)(s\exp X)=\int_{M(s)}\epsilon(s,\tau)e^{i\langle f,X\rangle}\beta_s \tag{3}$$

holds as an identity of locally L^1-functions on $\mathfrak{g}(s)$ [1]. Here $\epsilon(s,\tau)$ is a locally constant function on $M(s)$ (defined in [5]).

Recall (1) the measure $B_c(s,\Phi)$ on $\mathfrak{t}^*$ associated to s. Denote by $p_s: M(s)\to\mathfrak{h}(s)^*$ the restriction of p to $M(s)$. We define $DH(M,s,\tau)$ as the sum of the measures $\epsilon(s,\tau)_i DH(M_s^i,p_s^i)$, where M_s^i are the connected components of M_s, and $\epsilon(s,\tau)_i$ the constant value of $\epsilon(s,\tau)$ on M_s^i. The support of $DH(M,s,\tau)$ is contained in the image $p(M)$ of M for any s. Formula (3) implies the identity of measures on $\mathfrak{t}^*$:

$$\left(\sum_{\nu\in P_{\mathfrak{h}}} s^{\nu-\rho_H}m(\nu)\delta_\nu\right)*B_c(s,\Phi)=DH(M,s,\tau).$$

Comparing with Formula (2), we see that we can compute $m(\nu)$ from the knowledge, in a neighborhood of ν, of Duistermaat–Heckman measures $DH(M,s,\tau)$ associated to all vertices s. In particular $m(\nu)=0$, if ν is not in the interior $p(M)^0$ of $p(M)$.

More precisely, Theorem 2.2 and the definition of the quantization of (possibly singular) reduced spaces gives us

$$Q(M,\tau)|_H=\sum_{\nu\in P_{\mathfrak{h}}^+\cap p(M)^0} Q(M_\nu,\tau)Q(H\nu).$$

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