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A Note on time regularity of generalized Ornstein–Uhlenbeck processes with cylindrical stable noise

Une Note sur la régularité du temps des processus Ornstein–Uhlenbeck généralisée avec le bruit stables cylindriques

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ABSTRACT

A necessary and sufficient condition of càdlàg modification of Ornstein–Uhlenbeck process with cylindrical stable noise in a Hilbert space is given in this Note. Applying this result, some questions in *Time irregularity of generalized Ornstein–Uhlenbeck processes* [C. R. Acad. Sci. Paris, Ser. I 348 (2010) 273–276] and *Structural properties of semilinear SPDEs driven by cylindrical stable process* [Probab. Theory Related Fields 149 (2011) 97–137] are answered.

RÉSUMÉ

Une condition nécessaire et suffisante de la modification càdlàg des processus d'Ornstein–Uhlenbeck avec bruit stables cylindriques dans un éspace Hilbert est montré dans cette Note. En appliquant ce résultat, quelques questions dans *Time irregularity of generalized Ornstein–Uhlenbeck processes* [C. R. Acad. Sci. Paris, Ser. I 348 (2010) 273–276] et *Structural properties of semilinear SPDEs driven by cylindrical stable process* [Probab. Theory Related Fields 149 (2011) 97–137] sont répondues.

1. Introduction

The time regularity of stochastic process plays an important role in the study of some finer structural properties, such as (strong) Markov property, measurability and Doob's stopping time theorem (see [3,8]). In recent years, there are several substantial results on time regularity of SPDE driven by Lévy process (see [1,2,4,6–8] and references therein). In [1], and Section 4 in [8], the authors considered the càdlàg modification of Ornstein–Uhlenbeck process with Lévy noise on a Hilbert space H as follows,

$$\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + \mathrm{d}L(t), \quad t \geqslant 0. \tag{1}$$

Let A be the generator of a C_0-semigroup on H, and A^* be the adjoint operator of A. In [1], Brzeźniak et al. show that if the basis $\{e_n\}_{n\in\mathbb{N}}$ belongs to the domain $D(A^*)$, then $\beta_n \nrightarrow 0$ implies that H-valued process X in Eq. (1) has no H-càdlàg trajectory with probability 1 (see Theorem 2.1 in [1] for details). Then they present three questions on H-càdlàg modification

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naturally raised by the above result and a related question on stochastic heat equation with α-stable noise in [1]. If L^n are symmetric α-stable processes, $\alpha \in (0,2)$, Priola, Zabczyk conjectured in [8], under Hypothesis L (see Section 4 in [8] for details), the H-càdlàg property of Eq. (1) holds under much weaker conditions than $\sum_{n=1}^{\infty} \beta_n^{\alpha} < \infty$.

Inspired by [1,8], in this Note, we attempt to investigate the càdlàg modification of trajectories of Eq. (1). At first, we give a necessary condition of existence of H-càdlàg modification of Eq. (1) by the Lévy characteristic measure of L^n and β_n. This result implies Theorem 2.1 in [1] also (see Theorem 2.1 in Section 2 below). Then, applying Theorem 2.1, we show a necessary and sufficient condition of H-càdlàg modification of Eq. (1) driven by cylindrical α-stable processes (see Theorem 2.2 in Section 2). It seems that this result denies the conjecture in Subsection 4.1 in [8] and partly answers Question 3 in [1]. Furthermore, applying Theorem 2.2 to the stochastic heat equation (3) with α-stable noise, we answer Questions 1 and 4 in [1]. However, we are unable to answer Question 2 in [1] by the framework and method in this Note.

The main results are listed in Section 2 and the proofs are shown in Section 3.

2. Main results

In this Note, we focus on the same assumptions of solution $(X(t), t \geqslant 0)$ to Eq. (1) as in [1]. Let H be a separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$. Set L be a class of Lévy process with an expansion of the form $L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, where L^n are independent, identically distributed, càdlàg real-valued Lévy processes on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ with Lévy characteristic measure ν not being identically 0, $\{e_n\}_{n \in \mathbb{N}}$ is the fixed reference orthonormal basis in H and β_n is a sequence of positive numbers. Moreover, we assume $\{e_n\}_{n \in \mathbb{N}} \subset D(A^*)$, and the weak solution of Eq. (1) is represented as for each $n \in \mathbb{N}$,

$$\mathrm{d}\langle X(t), e_n \rangle_H = \langle X(t), A^* e_n \rangle_H \, \mathrm{d}t + \beta_n \, \mathrm{d}L^n(t).$$

Denote the processes $\langle X(t), e_n \rangle_H$ by $X^n(t)$. Then we have a necessary condition for $(X(t), t \geqslant 0)$ having H-càdlàg modification as follows:

Theorem 2.1. *Assume that the process X in Eq.* (1) *has H-càdlàg modification, then for any $\epsilon > 0$, $\sum_{n=1}^{\infty} \nu(|y| \geqslant \epsilon/\beta_n) < \infty$.*

Remark 1. Here, we emphasize that this theorem implies that if β_n does not converge to 0, then there exists ϵ such that $\sum_{n=1}^{\infty} \nu(|y| \geqslant \epsilon/\beta_n) = \infty$, hence X has no H-càdlàg modification. That is Theorem 2.1 in [1].

Then, we investigate the case when L^n, $n = 1, 2, \ldots$ are independent, identically distributed(i.i.d.), α-stable processes, $\alpha \in (0,2)$, of which the Lévy characteristic measure ν satisfies $\nu(\mathrm{d}y) = \begin{cases} c_1 y^{-1-\alpha}\,\mathrm{d}y, & y>0, \\ c_2 |y|^{-1-\alpha}\,\mathrm{d}y, & y<0, \end{cases}$ with $c_1 \geqslant 0$, $c_2 \geqslant 0$, $c_1 + c_2 > 0$ (see [9]). Therefore $\nu(|y| \geqslant \epsilon/\beta_n) = (c_1 + c_2)\beta_n^{\alpha}/(\alpha \epsilon^{\alpha})$ implies for any $\epsilon > 0$, $\sum_{n=1}^{\infty} \nu(|y| \geqslant |\epsilon/\beta_n|) < \infty$ if and only if $\sum_{n=1}^{\infty} |\beta_n|^{\alpha} < \infty$.

Theorem 4.13 in [6] says that the process L defines a Lévy process on H if and only if $\sum_{n=1}^{\infty} |\beta_n|^{\alpha} < \infty$. Moreover, as pointed out by Peszat, Zabczyk in Theorem 9.3 in [6], by Kotelenez regularity results if the C_0-semigroup $S(t)$ of A satisfies the so-called generalized contraction principle, i.e. $\|S(t)\|_{L(H,H)} \leqslant e^{\beta t}$ for all $t \geqslant 0$, where $\beta \geqslant 0$ is a constant then the property of L taking values in H ensures that X has H-càdlàg modification. So, if we assume that the generalized contraction principle of semigroup $S(t)$ holds, combining with Theorem 2.1, then we have

Theorem 2.2. *Assume $(L^n, n = 1, 2, \ldots)$ are i.i.d., non-trivial α-stable processes, $\alpha \in (0,2)$, then the following three assertions are equivalent*:

(1) *the process $(X(t), t \geqslant 0)$ in Eq.* (1) *has H-càdlàg modification*;
(2) $\sum_{n=1}^{\infty} |\beta_n|^{\alpha} < \infty$;
(3) *the process L is a Lévy process on H.*

Remark 2. In Subsection 4.1 in [8], considering $(L^n, n = 1, 2, \ldots)$ are i.i.d., non-trivial symmetric α-stable processes, Priola and Zabczyk conjectured that càdlàg modification property holds for Eq. (1) under much weaker condition than $\sum_{n=1}^{\infty} |\beta_n|^{\alpha} < \infty$. However, it seems that the above Theorem 2.2 denies their conjecture. Specially, Theorem 2.2 does not need the assumption of symmetry of L_n.

Remark 3. In [1], the authors have raised a question (refer to Question 3 in [1]) that whether the requirement that the process L evolves in H is also necessary for the existence of H-càdlàg modification of X or not. Theorem 2.2 answers this question partly, i.e. at least if $(L^n, n = 1, 2, \ldots)$ are independent, identically distributed, non-trivial α-stable processes, the process L evolving in H is a necessary condition of X having H-càdlàg modification.

Moreover, if the operator A in Eq. (1) is self-adjoint with eigenvectors e_n and the corresponding eigenvalues $-\lambda_n < 0$, $n \in \mathbb{N}$, then $(X^n(t), t \geqslant 0)$ is the $\mathbb{R}$-valued Ornstein–Uhlenbeck process defined by

$$\mathrm{d}X^n(t) = -\lambda_n X^n(t)\,\mathrm{d}t + \beta_n\,\mathrm{d}L^n(t), \quad t \geqslant 0. \tag{2}$$

For $\delta \in \mathbb{R}$, define $H_\delta = D(A^{\delta/2})$ with the naturally defined scalar product, i.e. $H_\delta = \{x = \sum_{n=1}^\infty x_n e_n\colon \sum_{n=1}^\infty \lambda_n^\delta |x_n|^2 < \infty, x_n \in \mathbb{R}\}$. Note $\{\lambda_n^{-\delta/2} e_n\}$ is an orthonormal basis in H_δ, as a direct corollary of Theorem 2.2, we have

Proposition 2.3. *Assume L^n are i.i.d., non-trivial α-stable processes, $\alpha \in (0,2)$ and X^n is the solution of Eq. (2). Then the following assertions are equivalent:*

(1) *the process $(X(t), t \geqslant 0)$ in Eq. (1) has H_δ-càdlàg modification;*
(2) $\sum_{n=1}^\infty |\beta_n \lambda_n^{\delta/2}|^\alpha < \infty$;
(3) *the process L is a Lévy process on H_δ.*

Furthermore, we apply Proposition 2.3 to the heat equation on $\mathcal{O} = (0,\pi)$ with α-stable noise introduced in [1] as follows:

$$\mathrm{d}X(t) = \Delta X(t)\,\mathrm{d}t + \mathrm{d}L(t), \tag{3}$$

where $Au = \Delta u$, $u \in D(A)(= H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))$ is the Laplacian Δ in $H(= L^2(\mathcal{O}))$ with Dirichlet boundary conditions. It is well known that $-\Delta$ is a self-adjoint operator with eigenfunctions $\{e_n(\cdot) = \sqrt{\frac{2}{\pi}}\sin(n\cdot)\}_{n\in\mathbb{N}}$ and eigenvalues $\{\lambda_n\colon \lambda_n = n^2\}_{n\in\mathbb{N}}$. Proposition 2.3 implies

Proposition 2.4. *If $\beta_n = 1$ for any $n \in \mathbb{N}$, Eq. (3) has H_δ-càdlàg modification if and only if $\delta < -1/\alpha$.*

Remark 4. Proposition 2.4 presents an answer to Question 4 in [1] that the process X in Eq. (3) has no H_δ-càdlàg modification for $\delta \in [-1/\alpha, 0)$.

Due to Proposition 4.2 in [8], if L^n are i.i.d., non-trivial symmetric α-stable processes, then $(X(t), t \geqslant 0)$ given in (3) takes value in H if and only if $\sum_{n=1}^\infty \beta_n^\alpha/n^2 < \infty$. So, setting $\delta = 0$ in Proposition 2.3 above, we have

Proposition 2.5. *Assume L^n are i.i.d., non-trivial symmetric α-stable processes. If $(\beta_n, n \geqslant 1)$ satisfies $\sum_{n=1}^\infty \beta_n^\alpha/n^2 < \infty$ and $\sum_{n=1}^\infty \beta_n^\alpha = \infty$, then there is no H-càdlàg modification of $(X(t), t \geqslant 0)$ in Eq. (3), even if for any $t > 0$, $X(t) \in H$.*

Remark 5. If we set $\beta_n = n^{-\frac{1}{\alpha}}$, then $\sum_{n=1}^\infty \beta_n^\alpha/n^2 < \infty$, $\sum_{n=1}^\infty \beta_n^\alpha = \infty$ and $\beta_n \to 0$ in Eq. (3). By Proposition 2.5, we give an example showing that $\beta_n \to 0$ does not imply the existence of H-càdlàg modification of X, even if for any $t > 0$, $X(t) \in H$ and the Lévy characteristic measure of L supports on H. This is a negative answer to Question 1 in [1].

3. Proofs

Before proving Theorem 2.1, we need some lemmas. Let $(X(t), t \geqslant 0)$ be an H-valued process on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ and $X^n(t) = \langle X(t), e_n\rangle_H$, where $\{e_n\}$ is an orthonormal basis in H. We have

Lemma 3.1. *The process $(X(t), t \geqslant 0)$ is an H-càdlàg (resp. continuous) process with probability 1 if and only if for any $n \in \mathbb{N}$, the process $(X^n(t), t \geqslant 0)$ is càdlàg (resp. continuous) process with probability 1 and for any $T > 0$,*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \sum_{i=N}^\infty \left|X^i(t)\right|^2 = 0, \quad \textit{with probability } 1. \tag{4}$$

Proof. ***The 'only if' part.*** Let $\Omega_0^{\hat{N}} = \{\omega \in \Omega\colon X(\omega)$ satisfies Eq. (4) for $T = \hat{N}\}$, $\hat{N} \in \mathbb{N}$, and $\Omega_0 = \bigcap_{\hat{N}=1}^\infty \Omega_0^{\hat{N}}(\bigcap_{n=1}^\infty \{\omega \in \Omega\colon X^n(\omega) \in D([0,\infty),\mathbb{R})\})$, then $\mathbb{P}(\Omega_0) = 1$. Fixing $\omega \in \Omega_0$ and $t \in [0,\infty)$, for any $\epsilon > 0$, by Eq. (4), there exists $N_{t,\omega,\epsilon} \in \mathbb{N}$ satisfying $\sup_{s\in[0,t+1]} \sum_{i=N_{t,\omega,\epsilon}}^\infty |X^i(s)|^2 \leqslant \epsilon$. Since the process $(X^n(t), t \geqslant 0)$ has the right-continuous property for any $n \in \mathbb{N}$ on Ω_0,

$$\limsup_{s'\downarrow t} \left\|X(s') - X(t)\right\|_H^2 \leqslant \lim_{s'\downarrow t} \sum_{i=1}^{N_{t,\omega,\epsilon}} \left|X^i(s') - X^i(t)\right|^2 + 2 \sup_{s\in[0,t+1]} \sum_{i=N_{t,\omega,\epsilon}}^\infty \left|X^i(s)\right|^2 \leqslant 2\epsilon. \tag{5}$$

On the other hand, similar to Eq. (5), for any $t > 0$,

$$\limsup_{s_1,s_2\uparrow t} \left\|X(s_1) - X(s_2)\right\|_H^2 \leqslant 2\epsilon, \tag{6}$$

because the process $(X^n(t), t \geqslant 0)$ has the left limits property for any $n \in \mathbb{N}$ on Ω_0.

Hence by Eq. (5) and Eq. (6), $(X(t), t \geqslant 0)$ is càdlàg in H on Ω_0.

The 'if' part. A well-known equivalent characterization of the compact set K in a separable Hilbert space V is that K is bounded, closed and for any orthonormal basis $\{v_n\}_{n\in\mathbb{N}}$ in V, for any $\epsilon > 0$, there is an $N_\epsilon \in \mathbb{N}$ (dependent on $\{v_n\}_{n\in\mathbb{N}}$), such that $\sup_{x\in K}\sum_{i=N_\epsilon}^{\infty}\langle x, v_i\rangle_V^2 < \epsilon$.

By Proposition 1.1 in [5], for any $x \in D([0,T], H)$, $\{x(t), t \in [0,T]\} \cup \{x(t-), t \in [0,T]\}$ is a compact set in H. Therefore, for a.e. $\omega \in \Omega$, using the above equivalent characterization of compact set in H, there is an $N_{T,\epsilon,\omega} \in \mathbb{N}$ such that $\sup_{t\in[0,T]}\sum_{i=N_{T,\epsilon,\omega}}^{\infty}|X^i(t)|^2 < \epsilon$.

Finally, since $|X^n(t) - X^n(s)|^2 \leqslant \|X(t) - X(s)\|_H^2$, the property that X is an H-càdlàg process with probability 1 implies $\mathbb{P}(\omega \in \Omega\colon\ X^n(\omega) \in D([0,\infty), \mathbb{R})) = 1$. □

Set $\triangle f(t) = f(t) - f(t-)$. Noting that if $(X(t), t \geqslant 0)$ is an H-càdlàg process, then $\sup_{n\geqslant N}\sup_{t\in[0,T]}|\triangle X^n(t)| \leqslant 2(\sup_{t\in[0,T]}\sum_{n=N}^{\infty}|X^n(t)|^2)^{1/2}$, hence by the 'only if' part of Lemma 3.1, we obtain

Lemma 3.2. *Assume the process $(X(t), t \geqslant 0)$ is an H-càdlàg process with probability* 1*, then for any $T > 0$,*

$$\lim_{N\to\infty}\sup_{n\geqslant N}\sup_{t\in[0,T]}\left|\triangle X^n(t)\right| = 0, \quad \textit{with probability } 1.$$

Proof of Theorem 2.1. Without loss of generality, we assume X has the H-càdlàg property. Let $\tau_n = \inf\{t > 0\colon |\beta_n \triangle L^n(t)| \geqslant \epsilon\}$, in particular $|\triangle \beta_n L^n(\tau_n)| \geqslant \epsilon$. Then τ_n, $n \in \mathbb{N}$ are mutually independent and exponentially distributed with parameter $\psi_n = \nu(|y| \geqslant \epsilon/\beta_n)$. Fixing $T > 0$, in view of the H-càdlàg property of X, Lemma 3.2 implies $\lim_{N\to\infty}\mathbb{P}(\tau_n \leqslant T, \text{for some } n \geqslant N) = 0$. Since $\mathbb{P}(\tau_n \leqslant T, \text{for some } n \geqslant N) = 1 - \prod_{n\geqslant N}\mathbb{P}(\tau_n \leqslant T) = 1 - \exp(-\sum_{n=N}^{\infty}\psi_n T)$, we get $\sum_{n=1}^{\infty}\nu(|y| \geqslant \epsilon/\beta_n) = \sum_{n=1}^{\infty}\psi_n < \infty$. □

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