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Simultaneous observability and stabilization of some uncoupled wave equations

Observabilité et stabilisation simultanées d'équations d'ondes découplées

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A R T I C L E   I N F O

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A B S T R A C T

We consider uncoupled wave equations with different speed of propagation in a bounded domain. Using a combination of the Bardos–Lebeau–Rauch observability result for a single wave equation and a new unique continuation result for uncoupled wave equations, we prove an observability estimate for that system. Applying Lions' Hilbert uniqueness method (HUM), one may derive simultaneous exact controllability results for the uncoupled system; the controls being locally distributed, with their supports satisfying the geometric control condition of Bardos, Lebeau and Rauch. Afterwards, we discuss the related simultaneous stabilization problem; this latter problem is solved by a combination of the new observability inequality, and a result of Haraux establishing an equivalence between observability and stabilization for second order evolution equations with bounded damping operators. Our observability and stabilization results generalize to higher space dimensions some earlier results of Haraux established in the one-dimensional setting.

R É S U M É

Nous considérons un système d'équations d'ondes découplées ayant des vitesses de propagation différentes dans un domaine borné. Utilisant une combinaison de l'inégalité d'observabilité de Bardos–Lebeau–Rauch pour une seule équation d'ondes ainsi qu'un nouveau résultat de continuation unique pour un système d'ondes découplées, nous démontrons une inégalité d'observabilité pour ce système. Une application de la méthode d'unicité de Hilbert de Lions (HUM), conduit, avec l'aide de cette inégalité à des résultats de contrôlabilité exacte simultanée ; les contrôles étant localement distribués dans le domaine en question, et leurs supports satisfaisant la propriété de contrôle géométrique de Bardos–Lebeau–Rauch. Par la suite, nous étudions le problème de stabilisation simultanée associé ; en particulier, nous démontrons la décroissance exponentielle de l'énergie avec l'aide de la nouvelle inégalité d'observabilité ainsi qu'un résultat de Haraux sur l'équivalence entre l'observabilité et la stabilisation d'équations d'évolution du second ordre en temps où l'opérateur décrivant l'amortissement est borné. Les résultats établis dans cette note généralisent à toutes dimensions d'espace certains résultas antérieurs de Haraux démontrés dans le cadre unidimensionel.

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On considère le système d'équations d'ondes découplées (3). Pour ce système, on a le résultat d'observabilité

Théorème 1. *Soit ω un ouvert satisfaisant la condition de contrôle géométrique (GCC) décrite ci-dessous. On suppose que les constantes a_j, $j = 1, 2, \ldots, q$, sont deux à deux distinctes. Soit $T > T_0 \max\{a_j^{-1/2};\ j = 1, 2, \ldots, q\}$, où T_0 est le temps de contrôlabilité dans* [3] *correspondant à l'équation des ondes ayant l'unité comme vitesse de propagation. Il existe une constante strictement positive $C_0 = C_0(\Omega, \omega, T, a)$, où $a = (a_j,\ j = 1, 2, \ldots, q)$, telle que l'inégalité d'observabilité suivante ait lieu*

$$E(0) \leqslant C_0 \int_0^T \int_\omega \left| \sum_{j=1}^q u_{j_t}(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t, \quad \textit{pour tout } \left(u_j^0, u_j^1\right)_j \in \left(H_0^1(\Omega) \times L^2(\Omega)\right)^q. \tag{1}$$

Par la suite, on considère le problème de stabilisation pour le système (7), et on a le

Théorème 2. *Soit $(y_j^0, y_j^1)_j \in (H_0^1(\Omega) \times L^2(\Omega))^q$. On suppose que les constantes a_j, $j = 1, 2, \ldots, q$, sont deux à deux distinctes.*

(i) *Soit $\mathcal{O}$ un ouvert non vide dans Ω. On suppose que l'amortissement est effectif dans $\mathcal{O}$. Alors l'énergie E vérifie $\lim_{t\to\infty} E(t) = 0$.*
(ii) *On suppose que $\mathcal{O}$ satisfait (GCC), et on suppose aussi que l'amortissement est effectif dans $\mathcal{O}$. Ils existent des constantes strictement positives $M = M(\Omega, \omega, T, a, d)$, et $\mu = \mu(\Omega, \omega, T, a, d)$ telles que l'estimation de décroissance de l'énergie suivante a lieu*

$$E(t) \leqslant M e^{-\mu t} E(0), \quad \textit{pour tout } t \geqslant 0. \tag{2}$$

1. Problem formulation and statements of main results

Let Ω be a bounded smooth open subset of $\mathbb{R}^N$, $N \geqslant 1$, and let T be a positive real number. Set $Q = \Omega \times (0, T)$. Let ω be a nonempty open subset of Ω, and let $a_1, a_2, \ldots, a_q$ ($q \geqslant 2$) be positive constants.

Consider the uncoupled wave equations

$$\begin{cases} u_{jtt} - a_j \Delta u_j = 0 \quad \text{in } Q, \\ u_j = 0 \quad \text{on } \Sigma = \partial\Omega \times (0, T), \\ u_j(x, 0) = u_j^0(x), \qquad u_{jt}(x, 0) = u_j^1(x), \quad \text{in } \Omega,\ j = 1, 2, \ldots, q, \end{cases} \tag{3}$$

where (u_j^0, u_j^1) lies in an appropriate Hilbert space for each j.

It is by now well-known that if (ω, T) satisfies the geometric control condition (GCC) of Bardos, Lebeau and Rauch [3]: every ray of geometric optics enters ω in a time less than T, then one has, for each j, the following observability inequality:

$$\|u_j^0\|_{H_0^1(\Omega)}^2 + \|u_j^1\|_{L^2(\Omega)}^2 \leqslant C \int_0^T \int_\omega \left|u_{jt}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t, \quad \forall \left(u_j^0, u_j^1\right) \in H_0^1(\Omega) \times L^2(\Omega), \tag{4}$$

where $C = C(\Omega, \omega, T, a_j)$ is a positive constant.

That estimate involves a single wave equation; its proof in the general case due to Bardos, Lebeau and Rauch uses microlocal analysis, and may be found in the Appendix of [10]. Other approaches for proving it are proposed in the literature: multipliers e.g. [8,10], differential geometry e.g. [9], Carleman estimates e.g. [5], etc. Although we now have a better picture on the controllability of a single wave equation, e.g. [3–6,8,10,9,12,15], the controllability of systems of wave equations is still at its infancy; the few results that exist in the literature involve special cases only, e.g. [1,6,10,11,13,14]. To the author's best knowledge, the only controllability results associated with (3) may be found in [6], where it is proven that exact controllability holds in the one-dimensional setting, or when the control region is the whole set Ω. The proofs in [6] use some linear algebra arguments combined with Fourier analysis. In [6], the author also shows that for every nonvoid open subset ω of Ω and a large enough time T, the quantity $\int_0^T \int_\omega |\sum_{j=1}^q u_j(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t$ defines a norm on $(L^2(\Omega) \times H^{-1}(\Omega))^q$, which is a unique continuation result; but the equivalence between that norm with the standard norm on $(L^2(\Omega) \times H^{-1}(\Omega))^q$ is not discussed therein. Our main purpose in this note is to show that an appropriate blend of the observability result known for a single wave equation [3,4], and a new unique continuation result for the uncoupled system (see Lemma 2.1 below) leads to

Theorem 1.1. *Assume that ω satisfies the geometric control condition described above. Suppose that the constants a_j, $j = 1, 2, \ldots, q$, are pairwise distinct. Let $T > T_0 \max\{a_j^{-1/2};\ j = 1, 2, \ldots, q\}$, where T_0 is the controllability time in* [3] *corresponding to the wave* *equation with unit speed of propagation. There exists a positive constant $C_0 = C_0(\Omega, \omega, T, a)$, with $a = (a_j,\ j = 1, 2, \dots, q)$, such that the following observability estimate holds*

$$E(0) \leqslant C_0 \int_0^T \int_\omega \left| \sum_{j=1}^q u_{j_t}(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t, \quad \textit{for all } \left(u_j^0, u_j^1\right)_j \in \left(H_0^1(\Omega) \times L^2(\Omega)\right)^q, \tag{5}$$

where

$$2E(0) = \sum_{j=1}^q \left(\|u_j^0\|^2_{H_0^1(\Omega)} + \|u_j^1\|^2_{L^2(\Omega)} \right).$$

Remark 1.2. Theorem 1.1 extends to the multidimensional setting the one-dimensional observability result [6,7]. It also generalizes to a system of uncoupled wave equations with different speeds of propagation the observability result known for a single wave equation e.g. [3,5,12].

Remark 1.3. It follows from Theorem 1.1

$$\widehat{E}(0) \leqslant C_0 \int_0^T \int_\omega \left| \sum_{j=1}^q u_j(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t, \quad \text{for all } \left(u_j^0, u_j^1\right)_j \in \left(L^2(\Omega) \times H^{-1}(\Omega)\right)^q, \tag{6}$$

where

$$2\widehat{E}(0) = \sum_{j=1}^q \left(\|u_j^0\|^2_{L^2(\Omega)} + \|u_j^1\|^2_{H^{-1}(\Omega)} \right).$$

Indeed, for $(u_j^0, u_j^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, the function φ_j given by $\varphi_j(x,t) = \int_0^t u_j(x,s)\,\mathrm{d}s + h_j(x)$, where $h_j \in H_0^1(\Omega)$ with $a_j \Delta h_j = u_j^1$, satisfies (3) with $\varphi(x,0) = h_j$, and $\varphi_{jt}(x,0) = u_j^0$. One then applies Theorem 1.1 to the φ_js to derive the claimed estimate.

Controllability results follow from Theorem 1.1, and (6) by the application of Lions' HUM.

The other problem that we want to tackle in this note is the simultaneous stabilization problem: Given $(y_j^0, y_j^1)_j \in (H_0^1(\Omega) \times L^2(\Omega))^q$, and a nonnegative bounded measurable function d defined on Ω, consider the damped system

$$\begin{cases} y_{jtt} - a_j \Delta y_j + d(x)(y_{1t} + y_{2t} + \dots + y_{qt}) = 0 & \text{in } \Omega \times (0, \infty), \\ y_j = 0 \quad \text{on } \partial\Omega \times (0, \infty), \\ y_j(x,0) = y_j^0(x), \qquad y_{jt}(x,0) = y_j^1(x), & \text{in } \Omega,\ j = 1, 2, \dots, q. \end{cases} \tag{7}$$

The total energy of (7) is given, for all $t \geqslant 0$, by

$$2E(t) = \sum_{j=1}^q \int_\Omega \left\{ \left| y_{jt}(x,t) \right|^2 + a_j \left| \nabla y(x,t) \right|^2 \right\} \mathrm{d}x, \tag{8}$$

and it is a nonincreasing function of the time variable as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int_\Omega d(x)(y_{1t} + y_{2t} + \dots + y_{qt})^2(x,t)\,\mathrm{d}x.$$

Let $\mathcal{O}$ be a nonvoid open subset in Ω, and suppose that the damping is effective in $\mathcal{O}$, viz.: $\exists a_0 > 0$: $d(x) \geqslant a_0$ a.e. in $\mathcal{O}$. The two questions that we would like to answer are the following:

- Does the energy $E(t)$ decays to zero as $t \to \infty$?
- If $\mathcal{O}$ satisfies (GCC), do we have a uniform exponential decay of $E(t)$ in the energy space?

The answers to those two questions are provided by

Theorem 1.4. *Let $(y_j^0, y_j^1)_j \in (H_0^1(\Omega) \times L^2(\Omega))^q$. Suppose that the constants a_j, $j = 1, 2, \dots, q$, are pairwise distinct.*

(i) *Let $\mathcal{O}$ be a nonvoid open subset in Ω, and suppose that the damping is effective in $\mathcal{O}$. Then the energy E satisfies* $\lim_{t\to\infty} E(t) = 0$.

(ii) *Assume that $\mathcal{O}$ satisfies (GCC), and suppose that the damping is effective in $\mathcal{O}$. There exist positive constants $M = M(\Omega, \omega, T, a, d)$, and $\mu = \mu(\Omega, \omega, T, a, d)$ such that the following energy decay estimate holds*

$$E(t) \leqslant Me^{-\mu t}E(0), \quad \text{for all } t \geqslant 0. \tag{9}$$

2. Sketch of the proof of Theorem 1.1

Let ω_0 be another open subset in Ω which satisfies (GCC), and such that its closure lies entirely in ω. Introduce the function η, which satisfies:

$$\eta \in C^1(\bar{\Omega}), \quad 0 \leqslant \eta \leqslant 1, \qquad \eta = 1 \quad \text{in } \omega_0, \qquad \eta = 0 \quad \text{in } \Omega \setminus \omega. \tag{10}$$

Let $\alpha > 0$ be a constant with $T - 2\alpha > T_0 \max\{a_j^{-1/2};\ j = 1, 2, \dots, q\}$. Let $r \in \mathcal{C}^1([0, T])$ with

$$0 \leqslant r \leqslant 1, \qquad r(0) = r(T) = 0, \qquad r \equiv 1 \quad \text{on } [\alpha, T - \alpha]. \tag{11}$$

Thanks to (4), (10), (11), and the fact that the wave equation is translation invariant, we have

$$E(0) \leqslant C_0 \int_\alpha^{T-\alpha} \int_{\omega_0} \sum_{j=1}^q \left|u_{jt}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \leqslant C_0 \int_0^T r(t) \int_\omega \eta(x) \sum_{j=1}^q \left|u_{jt}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t, \tag{12}$$

where here and in the sequel C_0 is a generic constant having the form stated in the theorem.

Now

$$C_0 \int_0^T r(t) \int_\omega \eta(x) \sum_{j=1}^q \left|u_{jt}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t = C_0 \int_0^T r(t) \int_\omega \eta(x) \left|\sum_{j=1}^q u_{jt}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t$$
$$- 2C_0 \int_0^T r(t) \int_\omega \eta(x) \sum_{1\leqslant j<k\leqslant q} u_{jt}(x,t) u_{kt}(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{13}$$

Reporting (13) in (12), and using (10) and (11), one finds

$$E(0) \leqslant C_0 \int_0^T \int_\omega \left|\sum_{j=1}^q u_{jt}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t - 2C_0 \int_0^T r(t) \int_\omega \eta(x) \sum_{1\leqslant j<k\leqslant q} u_{jt}(x,t) u_{kt}(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{14}$$

It remains to absorb the utmost right term in (14); the main obstacle is that the cross-products involve first order terms only. We are going to transform that term so that the cross-products involve first and zero order terms. Once that is done, one can use Young's inequality to absorb all first order terms, and a compactness argument to absorb the zero order terms.

Let $1 \leqslant j < k \leqslant q$. Multiply the j-th equation by $r\eta a_k u_k$, and the k-th equation by $r\eta a_j u_j$, and integrate by parts on Q to obtain respectively

$$-a_k \int_Q r'\eta u_{jt} u_k\,\mathrm{d}x\,\mathrm{d}t - a_k \int_Q r\eta u_{jt} u_{kt}\,\mathrm{d}x\,\mathrm{d}t + a_k a_j \int_Q r\eta(\nabla u_j \cdot \nabla u_k)\,\mathrm{d}x\,\mathrm{d}t + a_k a_j \int_Q r u_k(\nabla u_j \cdot \nabla \eta)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{15}$$

and

$$-a_j \int_Q r'\eta u_{kt} u_j\,\mathrm{d}x\,\mathrm{d}t - a_j \int_Q r\eta u_{jt} u_{kt}\,\mathrm{d}x\,\mathrm{d}t + a_k a_j \int_Q r\eta(\nabla u_j \cdot \nabla u_k)\,\mathrm{d}x\,\mathrm{d}t + a_k a_j \int_Q r u_j(\nabla u_k \cdot \nabla \eta)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{16}$$

Subtracting (16) from (15) side by side, we find

$$(a_j - a_k) \int_Q r\eta u_{jt} u_{kt}\,\mathrm{d}x\,\mathrm{d}t = \int_Q r'\eta(a_k u_{jt} u_k - a_j u_{kt} u_j) - a_k a_j r(u_k \nabla u_j - u_j \nabla u_k) \cdot \nabla\eta\,\mathrm{d}x\,\mathrm{d}t. \tag{17}$$

Reporting (17) in (14), we get, keeping in mind that $a_j \neq a_k$ for $j \neq k$

$$E(0) \leqslant C_0 \int_0^T \int_\omega \left|\sum_{j=1}^q u_{jt}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t$$
$$- 2C_0 \sum_{1\leqslant j<k\leqslant q} \frac{1}{a_j - a_k} \int_Q r'\eta(a_k u_{jt} u_k - a_j u_{kt} u_j) - a_k a_j r(u_k \nabla u_j - u_j \nabla u_k) \cdot \nabla\eta\,\mathrm{d}x\,\mathrm{d}t. \tag{18}$$

The application of Young's inequality to the term in the lower line in (18), one easily derives for each $\varepsilon > 0$

$$E(0) \leqslant C_0 \int_0^T \int_\omega \left| \sum_{j=1}^q u_{jt}(x,t) \right|^2 dx\,dt + \varepsilon \int_Q \sum_{j=1}^q \left(\left| u_{jt}(x,t) \right|^2 + \left| \nabla u_j(x,t) \right|^2 \right) dx\,dt + C_\varepsilon \int_Q \sum_{j=1}^q \left| u_j(x,t) \right|^2 dx\,dt. \tag{19}$$

Since the energy is constant, we have

$$\int_Q \sum_{j=1}^q \left(\left| u_{jt}(x,t) \right|^2 + a_j \left| \nabla u_j(x,t) \right|^2 \right) dx\,dt = 2T E(0). \tag{20}$$

Combining (19) and (20), and choosing $\varepsilon = 1/4T$, we find after some algebra

$$E(0) \leqslant C_0 \int_0^T \int_\omega \left| \sum_{j=1}^q u_{jt}(x,t) \right|^2 dx\,dt + C_0 \int_Q \sum_{j=1}^q \left| u_j(x,t) \right|^2 dx\,dt. \tag{21}$$

Now we are going to use the unique continuation result mentioned above to absorb the utmost right term in (21). To this end, we claim

$$\int_Q \sum_{j=1}^q \left| u_j(x,t) \right|^2 dx\,dt \leqslant C_0 \int_0^T \int_\omega \left| \sum_{j=1}^q u_{jt}(x,t) \right|^2 dx\,dt, \quad \forall \left(u_j^0, u_j^1 \right)_j \in \left(H_0^1(\Omega) \times L^2(\Omega) \right)^q. \tag{22}$$

That claim is proven by contradiction. Suppose that (22) fails. Then one can show that there are initial data in $(H_0^1(\Omega) \times L^2(\Omega))^q$ for which

$$\int_Q \sum_{j=1}^q \left| u_j(x,t) \right|^2 dx\,dt = 1, \qquad \sum_{j=1}^q u_{jt}(x,t) = 0 \quad \text{in } \omega \times (0,T). \tag{23}$$

The contradiction follows from

Lemma 2.1. *Let ω be an arbitrary nonvoid open subset in Ω. Let T, the constants a_js, and the initial data be given as in Theorem* 1.1. *Then*

$$\sum_{j=1}^q u_{jt}(x,t) = 0 \quad \textit{in } \omega \times (0,T)$$

implies $u_j \equiv 0$ in Q.

Lemma 2.1 may be proven following the proof of Proposition 2.1.1 in [6]. This completes the proof sketch of Theorem 1.1. □

Sketch of the proof of Theorem 1.4. (i) If one denotes by $\mathcal{A}$ the underlying unbounded operator, then $\mathcal{A}$ generates a C_0 semigroup of contractions $(S(t)_{t \geqslant 0})$ on $\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^q$. Further, $\mathcal{A}$ has a compact resolvent; so the spectrum $\sigma(\mathcal{A})$ is discrete. Adapting the approach in [6] for the proof of Proposition 2.1.1, one shows that $\mathcal{A}$ has no purely imaginary eigenvalue. The stability theorem in [2] shows the claimed strong stability result.

(ii) Thanks to Theorem 1.1 above, and Proposition 1 in [7], which establishes an equivalence between observability and stabilization for second order evolution equations with bounded damping operators, the claimed exponential decay follows. □

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