C. R. Acad. Sci. Paris, Ser. I 350 (2012) 45–50

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial Differential Equations

Existence of weak solutions to a simplified steady system of turbulence modeling

Existence des solutions faibles pour un système stationaire simplifié de turbulence

Joachim Naumann [a], Joerg Wolf [b]

[a] *Department of Mathematics, Humboldt University Berlin, Unter den Linden 6, 10099 Berlin, Germany*
[b] *Faculty of Mathematics, University of Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany*

ARTICLE INFO

Article history:
Received 26 July 2011
Accepted 12 December 2011
Available online 21 December 2011

Presented by Philippe G. Ciarlet

ABSTRACT

We consider a coupled system of PDEs for two scalar functions u and k in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) of Prandtl's (1945) turbulence model (u = "one-dimensional" mean velocity, k = turbulent mean kinetic energy). We prove the existence of weak solutions to the system under consideration with homogeneous Dirichlet conditions on u, and mixed boundary conditions on k.

RÉSUMÉ

On considère un système couplé d'équations aux dérivées partielles pour des fonctions scalaires u et k dans un domaine borné de $\mathbb{R}^d$ ($d = 2$ ou $d = 3$). Ce système représente une version simplifiée du modèle stationaire de turbulence de Prandtl (1945) (u = vitesse « unidimensionnelle » moyenne, k = énergie cinétique turbulente moyenne). On établit l'existence des solutions faibles du système envisagé avec des conditions aux limites homogènes de Dirichlet pour u, et des conditions aux limites mixtes homogènes de Neumann pour k.

Version française abrégée

Dans un domaine borné de Lipschitz $\Omega \subset \mathbb{R}^d$ ($d = 2$ ou $d = 3$) on étudie le système

$$-\operatorname{div}\big((\nu + \sqrt{k})\nabla u\big) = f, \qquad -\operatorname{div}\big(\sqrt{k}\nabla k\big) = \sqrt{k}|\nabla u|^2 - k\sqrt{k}. \tag{1}$$

Ce système représente une version simplifiée du modèle stationaire de turbulence de Prandtl (voir [4,7,8]). On considère les conditions aux limites suivantes pour u et k :

$$u = 0 \quad \text{sur } \partial\Omega, \tag{2a}$$

$$k = k_D \quad \text{sur } \Gamma_D, \qquad \sqrt{k}\frac{\partial k}{\partial \mathbf{n}} = 0 \quad \text{sur } \Gamma_N \tag{2b}$$

où $\partial\Omega = \Gamma_D \cup \Gamma_N$ disjoint, avec $\Gamma_D \neq \emptyset$ relativement ouverte.

E-mail address: jnaumann@mathematik.hu-berlin.de (J. Naumann).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.12.008

Le but de cette Note est d'établir l'existence des solutions faibles du problème (1), (2a), (2b) de sorte que $k \geqslant \sigma k_D$ $(0 < \sigma < 1)$ p.p. dans Ω si $k_D = \text{const} > 0$ sur Γ_D. Si $k_D = 0$ sur Γ_D, on obtient $k \geqslant 0$ p.p. dans Ω, et il existe un ensemble $\Omega^* \subset \Omega$ de mesure positive de sorte que $k > 0$ p.p. dans Ω^*.

Les démonstrations procèdent en trois étapes. Si $k_D = \text{const} > 0$ sur Γ_D on prouve l'existence d'une solution faible du système régularisé

$$-\operatorname{div}\big(\big(\nu + [h]_\varepsilon^{1/3}\big)\nabla u\big) = f \quad \text{dans } \Omega, \qquad -\frac{2}{3}\Delta h + h = [h]_\varepsilon^{1/3}\frac{|\nabla u|^2}{1+\varepsilon|\nabla u|^2} \quad \text{dans } \Omega$$

avec les conditions aux limites (2a), et $h = k_D^{3/2}$ sur Γ_D, $\frac{\partial h}{\partial n} = 0$ sur Γ_N, où $[\xi]_\varepsilon := \min\{\frac{1}{\varepsilon}, \xi\}$ $(0 \leqslant \xi < +\infty, \varepsilon > 0)$. Ensuite on établit des estimations a-priori pour la solution faible $(u_\varepsilon, h_\varepsilon)$ du problème ci-dessus (par ex., $h_\varepsilon \geqslant (\sigma k_D)^{3/2}$ p.p. dans Ω à l'aide du principe du maximum pour la function $(-h_\varepsilon)$ avec $0 < \sigma < 1$ convenable) et on effectue le passage à la limite $\varepsilon \to 0$ pour $(u_\varepsilon, k_\varepsilon)$ où $k_\varepsilon = h_\varepsilon^{2/3}$. Au cas où $k_D = 0$ sur Γ_D on considère une modification du système régularisé ci-dessus.

Dans un travail à venir nous étudierons le cas instationaire de (1), (2a), (2b).

1. Statement of the problem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2$ or $d = 3)$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the following system of PDEs:

$$-\operatorname{div}\big(\big(\nu + \sqrt{k}\big)\nabla u\big) = f \quad \text{in } \Omega, \tag{1a}$$

$$-\operatorname{div}\big(\sqrt{k}\nabla k\big) = \sqrt{k}|\nabla u|^2 - k\sqrt{k} \quad \text{in } \Omega. \tag{1b}$$

This system can be regarded as a simplified stationary version of Prandtl's one equation model of turbulence [8] (see, e.g., [4,7] for more details). Here, u can be viewed as the "one-dimensional" mean velocity of the flow, and k as the turbulent mean kinetic energy ($\nu = \text{const} > 0$, $\sqrt{k} = $ eddy viscosity).

Let be $\partial\Omega = \Gamma_D \cup \Gamma_N$ disjoint, where $\Gamma_D \neq \emptyset$ and relatively open. Let $\mathbf{n}$ denote the unit outward normal along $\partial\Omega$. We consider system (1a), (1b) with the following boundary conditions:

$$u = 0 \quad \text{on } \partial\Omega, \tag{2a}$$

$$k = k_D \quad \text{on } \Gamma_D, \qquad \sqrt{k}\frac{\partial k}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_N, \tag{2b}$$

where $k_D \geqslant 0$ is a given function.

System (1a), (1b) [without the term $k\sqrt{k}$] and with general unbounded coefficients in place of $\sqrt{k}$, has been studied in [3]. However, the conditions on the coefficients considered in this paper, exclude the physically important case $\sqrt{k}$. In [1], the authors investigated a similar system with unbounded coefficients. The full RANS model (with mean velocity $\mathbf{u} = (u_1, \dots, u_d)$, $\operatorname{div}\mathbf{u} = 0$) with certain restrictions on the coefficients (eddy viscosities) is studied in [2] and [5].

The aim of our paper is to present some existence results for weak solutions (u, k) to (1a), (1b), (2a), (2b) such that $k > 0$ a.e. at least on a set $\Omega^* \subset \Omega$ with $\operatorname{mes}\Omega^* > 0$. More specifically, if $k_D > 0$ on Γ_D, we show that $k \geqslant \text{const} > 0$ a.e. in Ω provided $\operatorname{mes}\Omega$ satisfies a smallness condition. If, however, $k_D = 0$ on Γ_D then $k \equiv 0$ ("laminar freestream", cf. [8, p. 11]) is a solution to (1b) whenever $|\nabla u| < +\infty$ in Ω. To motivate our discussion below, we suppose that there exists a sufficiently regular classical solution (u, k) to (1a), (1b), (2a), (2b) such that $k > 0$ in Ω. In (1b) we calculate $\operatorname{div}(\sqrt{k}\nabla k)$, then divide each term of this equation by $\sqrt{k}$ and multiply the new equation by $\psi \in C_c^2(\Omega)$, $\psi \geqslant 0$ in Ω. Integration by parts of the term $(-\Delta k)\psi$ gives

$$\int_\Omega k(-\Delta\psi + \psi) = \int_\Omega \left(\frac{|\nabla k|^2}{2k} + |\nabla u|^2\right)\psi \geqslant \int_\Omega |\nabla u|^2\psi.$$

Conversely, Theorem 2 states the existence of a weak solution (u, k) $(k \geqslant 0$ a.e. in $\Omega)$ to (1a), (1b), (2a), (2b) that satisfies

$$\int_\Omega k(-\Delta\psi + \psi) \geqslant \int_\Omega |\nabla u|^2\psi \quad \forall\psi \in C_c^2(\Omega),\ \psi \geqslant 0 \text{ in } \Omega$$

(cf. (9) below). Thus, if $\int_\Omega |\nabla u|^2 > 0$, then a standard argument shows that there exists a measurable set $\Omega^* \subset \Omega$ such that $\operatorname{mes}\Omega^* > 0$ and $k > 0$ a.e. in Ω^*.

2. Main results

Let $W^{m,p}(\Omega)$ $(m \in \mathbb{N},\ p \in [1, +\infty])$ denote the usual Sobolev space. If $\Gamma_D \neq \emptyset$ and relatively open, we define

$$W_{\Gamma_D}^{1,p}(\Omega) := \big\{v \in W^{1,p}(\Omega);\ v = 0 \text{ a.e. on } \Gamma_D\big\}.$$

It is well known that there exists $\gamma_0 = \gamma_0(\Omega) > 0$ such that

$$\|z\|_{L^6(\Omega)} \leqslant \gamma_0 \|\nabla z\|_{L^2(\Omega)} \quad \forall z \in W^{1,2}_{\Gamma_D}(\Omega) \ (d = 2 \text{ resp. } d = 3).$$

If $\Gamma_D = \partial\Omega$, we write $W^{1,p}_0(\Omega)$ in place of $W^{1,p}_{\partial\Omega}(\Omega)$.

Without any further reference, throughout we assume that $f \in L^r(\Omega)$ $(r > \frac{d}{2})$ and $k_D = \text{const} \geqslant 0$.

Theorem 1. *Let $k_D > 0$. Suppose that $\gamma_0^2 (\operatorname{mes} \Omega)^{2/3} < \frac{1}{3 \cdot 2^{3/2}}$. Then there exist a pair*

$$(u, k) \in \big(W^{1,2}_0(\Omega) \cap L^\infty(\Omega)\big) \times \bigcap_{1 \leqslant p < \frac{d}{d-1}} W^{1,p}(\Omega) \quad \textit{and} \quad \sigma \in]0, 1[$$

such that $k \geqslant \sigma k_D$ a.e. in Ω, $k = k_D$ a.e. on Γ_D,

$$\nabla\big(k^{3/2}\big) \in \bigcap_{1 \leqslant p < \frac{d}{d-1}} \mathbf{L}^p(\Omega), \quad k^{1/4} \nabla u, \nabla k^{1/4} \in \mathbf{L}^2(\Omega), \qquad \int_\Omega \frac{|\nabla k|^2}{k^{(1+2\delta)/2}} \leqslant c(\delta) \quad \forall 0 < \delta < 1, \tag{3}$$

where $c(\delta) \to +\infty$ as $\delta \to 0$, and

$$\int_\Omega \big(\nu + k^{1/2}\big) \nabla u \cdot \nabla \zeta = \int_\Omega f \zeta \quad \forall \zeta \in C_c^\infty(\Omega), \qquad \int_\Omega \big(\nu + k^{1/2}\big) |\nabla u|^2 = \int_\Omega f u \quad (\textit{energy identity}), \tag{4}$$

$$\int_\Omega k^{1/2} \nabla k \cdot \nabla \varphi = \int_\Omega \big(k^{1/2} |\nabla u|^2 - k^{3/2}\big) \varphi \quad \forall \varphi \in \bigcup_{s > d} W^{1,s}_{\Gamma_D}(\Omega). \tag{5}$$

Theorem 2. *Let $k_D = 0$. Then there exists a pair*

$$(u, h) \in \big(W^{1,2}_0(\Omega) \cap L^\infty(\Omega)\big) \times \bigcap_{1 \leqslant p < \frac{d}{d-1}} W^{1,p}_{\Gamma_D}(\Omega)$$

such that $h \geqslant 0$ a.e. in Ω, and

$$h^{1/6} \nabla u \in \mathbf{L}^2(\Omega), \quad \int_\Omega \frac{|\nabla h|^2}{(\lambda + h)^{1+\delta}} \leqslant \frac{c}{\delta \lambda^\delta} \quad \forall \lambda > 0, \ \forall \delta > 0 \ (c = \text{const}), \tag{6}$$

$$\frac{2}{3} \int_\Omega \nabla h \cdot \nabla \varphi = \int_\Omega \big(h^{1/3} |\nabla u|^2 - h\big) \varphi \quad \forall \varphi \in \bigcup_{s > d} W^{1,s}_{\Gamma_D}(\Omega). \tag{7}$$

Define $k := h^{2/3}$. Then $k \geqslant 0$ a.e. in Ω, $k \in \bigcap_{1 \leqslant p < p_0} L^p(\Omega)$ ($p_0 = +\infty$ if $d = 2$, $p_0 = \frac{9}{2}$ if $d = 3$), and the pair (u, k) satisfies

$$\int_\Omega \big(\nu + k^{1/2}\big) \nabla u \cdot \nabla \zeta = \int_\Omega f \zeta \quad \forall \zeta \in C_c^\infty(\Omega), \qquad \int_\Omega \big(\nu + k^{1/2}\big) |\nabla u|^2 = \int_\Omega f u \quad (\textit{energy identity}). \tag{8}$$

In addition,

$$\int_\Omega k(-\Delta \psi + \psi) \geqslant \int_\Omega |\nabla u|^2 \psi \quad \forall \psi \in C_c^\infty(\Omega), \ \psi \geqslant 0 \text{ in } \Omega. \tag{9}$$

3. Proof of Theorem 1

For $\xi \in [0, +\infty[$, $\varepsilon \in]0, 1[$, define $[\xi]_\varepsilon := \min\{\frac{1}{\varepsilon}, \xi\}$. We consider the weak formulation of (1a), (1b), (2a), (2b) with the new unknown $h := k^{3/2}$ instead of k $(\geqslant 0)$ and replace $h^{1/3}$ by $[h]_\varepsilon^{1/3}$. Set $h_D := k_D^{3/2}$. For every $\varepsilon > 0$ there exists $(u_\varepsilon, h_\varepsilon) \in W^{1,2}_0(\Omega) \times W^{1,2}(\Omega)$ such that

$$h_\varepsilon \geqslant 0 \quad \text{a.e. in } \Omega, \qquad h_\varepsilon = h_D \quad \text{a.e. on } \Gamma_D,$$

$$\int_\Omega \big(\nu + [h_\varepsilon]_\varepsilon^{1/3}\big) \nabla u_\varepsilon \cdot \nabla v = \int_\Omega f v \quad \forall v \in W^{1,2}_0(\Omega), \tag{10}$$

$$\frac{2}{3} \int_\Omega \nabla h_\varepsilon \cdot \nabla \varphi + \int_\Omega h_\varepsilon \varphi = \int_\Omega [h_\varepsilon]_\varepsilon^{1/3} \frac{|\nabla u_\varepsilon|^2}{1 + \varepsilon |\nabla u_\varepsilon|^2} \varphi \quad \forall \varphi \in W^{1,2}_{\Gamma_D}(\Omega). \tag{11}$$

This can be easily seen when replacing (10), (11) by an equivalent operator equation in the Hilbert space $W_0^{1,2}(\Omega) \times W_{\Gamma_D}^{1,2}(\Omega)$ and applying the theory of pseudo-monotone operators (see, e.g., [10, Chap. 27.3]; details of this argument are carried out in [6] for a problem which is similar to (1a), (1b), (2a), (2b)). From (10) one easily derives an L^1-estimate on $(\nu + [h_\varepsilon]_\varepsilon)^{1/3}|\nabla u_\varepsilon|^2$ as well as an L^∞-estimate on u_ε (see [9] for L^∞-estimates and maximum principles for a large class of elliptic equations in divergence form).

Define $w_\varepsilon := -h_\varepsilon$ a.e. in Ω. We multiply (11) by (-1) and take $\varphi = (w_\varepsilon - \lambda)^+$ $(\lambda \geqslant \lambda_0 := -h_D)$. This gives

$$\int_\Omega \left|\nabla(w_\varepsilon - \lambda)^+\right|^2 \leqslant \frac{3}{2} \int_{\{w_\varepsilon > \lambda\}} h_\varepsilon(w_\varepsilon - \lambda) \leqslant \frac{3}{2} h_D \int_\Omega (w_\varepsilon - \lambda)^+.$$

Again appealing to [9], we conclude that

$$w_\varepsilon(x) \leqslant \lambda_0 + 2^{\tau/(\tau-1)} \gamma_0^2 (\operatorname{mes}\Omega)^{\tau-1} \cdot \frac{3}{2} h_D (\operatorname{mes}\Omega)^{1/q} \quad \text{for a.e. } x \in \Omega,$$

where $\tau := 2(1 - \frac{1}{2^*}) - \frac{1}{q}$, $2^* = 6$ for both $d = 2$ and $d = 3$, and $q \in]1, +\infty[$ if $d = 2$, $q \in]\frac{3}{2}, +\infty[$ if $d = 3$. Hence

$$h_\varepsilon(x) \geqslant h_D - 3 \cdot 2^{1/(\tau-1)} \gamma_0^2 (\operatorname{mes}\Omega)^{\tau-1+1/q} h_D \quad \text{for a.e. } x \in \Omega.$$

Observing the smallness condition upon $\gamma_0^2(\operatorname{mes}\Omega)^{2/3}$, we fix a sufficiently large q such that $3 \cdot 2^{3q/(2q-3)} \gamma_0^2 (\operatorname{mes}\Omega)^{2/3} < 1$. Then there exists $\sigma \in]0, 1[$ such that $h_\varepsilon(x) \geqslant \sigma^{3/2} h_D$ for a.e. $x \in \Omega$.

Next, inserting $\varphi = 1 - \frac{h_D^\delta}{h_\varepsilon^\delta}$ $(0 < \delta < 1)$ into (11) and observing that

$$\int_\Omega h_\varepsilon \left(1 - \frac{h_D^\delta}{h_\varepsilon^\delta}\right) \geqslant -c \quad (c = \text{const} > 0 \text{ independent of } \varepsilon),$$

we obtain

$$\int_\Omega \frac{|\nabla h_\varepsilon|^2}{h_\varepsilon^{1+\delta}} \leqslant c(\delta) \quad \forall 0 < \delta < 1 \quad \big(c(\delta) \to +\infty \quad \text{as } \delta \to 0\big). \tag{12}$$

We now define $k_\varepsilon := h_\varepsilon^{2/3}$ a.e. in Ω (notice $[h_\varepsilon]_\varepsilon^{1/3} = [k_\varepsilon]_{\varepsilon^{2/3}}^{1/2}$), and rewrite (10), (11) for the pair $(u_\varepsilon, k_\varepsilon)$. Then from (12) it follows that, for all $\varepsilon > 0$ and all $1 \leqslant p < \frac{d}{d-1}$,

$$\|k_\varepsilon\|_{W^{1,p}(\Omega)}^p + \int_\Omega \left|k_\varepsilon^{1/2} \nabla k_\varepsilon\right|^p \leqslant C_1, \qquad \int_\Omega \left|\nabla k_\varepsilon^{1/4}\right|^2 \leqslant C_2 \quad \left(C_1 \to +\infty \quad \text{as } p \to \frac{d}{d-1}\right).$$

The estimates on $(u_\varepsilon, k_\varepsilon)$ imply the existence of a subsequence (not relabelled) such that $(u_\varepsilon, k_\varepsilon) \to (u, k)$ weakly in $W_0^{1,2}(\Omega) \times W^{1,p}(\Omega)$ $(1 < p < \frac{d}{d-1})$, a.e. in Ω and a.e. on $\partial\Omega$ as $\varepsilon \to 0$. Clearly, $k \geqslant \sigma k_D$ a.e. in Ω and $k = k_D$ a.e. on Γ_D. The passage to the limit $\varepsilon \to 0$ in (10) (with $k_\varepsilon = h_\varepsilon^{2/3}$) gives the first integral relation in (4), while the energy identity in (4) can be proved by a straightforward modification of an approximation argument developed in [3]. Finally, with the help of the energy identity in (4) one easily shows that $[k_\varepsilon]_{\varepsilon^{2/3}}^{1/2} |\nabla u_\varepsilon|^2 (1 + \varepsilon|\nabla u_\varepsilon|^2)^{-1} \to k^{1/2}|\nabla u|^2$ strongly in $L^1(\Omega)$ as $\varepsilon \to 0$. Then the passage to the limit $\varepsilon \to 0$ in (11) gives (5).

4. Proof of Theorem 2

In contrast to the proof of Theorem 1, we now use a double approximation procedure. This will enable us to prove (9).

For every $\varepsilon > 0$, $\eta > 0$ there exists $(u_{\varepsilon,\eta}, h_{\varepsilon,\eta}) \in W_0^{1,2}(\Omega) \times W_{\Gamma_D}^{1,2}(\Omega)$ such that $h_{\varepsilon,\eta} \geqslant 0$ a.e. in Ω, $h_{\varepsilon,\eta} = 0$ a.e. on Γ_D and

$$\int_\Omega \left(\nu + \big(\varepsilon + [h_{\varepsilon,\eta}]_\eta\big)^{1/3}\right) \nabla u_{\varepsilon,\eta} \cdot \nabla v = \int_\Omega f v \quad \forall v \in W_0^{1,2}(\Omega), \tag{13}$$

$$\frac{2}{3} \int_\Omega \nabla h_{\varepsilon,\eta} \cdot \nabla\varphi + \int_\Omega h_{\varepsilon,\eta}\varphi = \int_\Omega \big(\varepsilon + [h_{\varepsilon,\eta}]_\eta\big)^{1/3} \frac{|\nabla u_{\varepsilon,\eta}|^2}{1 + \eta|\nabla u_{\varepsilon,\eta}|^2} \varphi \quad \forall \varphi \in W_{\Gamma_D}^{1,2}(\Omega). \tag{14}$$

With minor notational alterations, this existence result can be proved by the same arguments as in the proof of Theorem 1. As above, from (13) we obtain an L^1-estimate on $(\nu + (\varepsilon + [h_{\varepsilon,\eta}]_\eta)^{1/3})|\nabla u_{\varepsilon,\eta}|^2$ and an L^∞-estimate on u_ε. Next, we insert $\varphi = 1 - \frac{\lambda^\delta}{(\lambda + h_{\varepsilon,\eta})^\delta}$ $(\lambda > 0,\ \delta > 0)$ into (14) to obtain

$$\int_\Omega \frac{|\nabla h_{\varepsilon,\eta}|^2}{(\lambda+h_{\varepsilon,\eta})^{1+\delta}} \leqslant \frac{c}{\delta\lambda^\delta} \quad \forall\lambda>0,\ \forall\delta>0\ \ (c=\text{const independent of } \varepsilon,\eta). \tag{15}$$

As above, this estimate implies $\|h_{\varepsilon,\eta}\|_{W^{1,p}(\Omega)} \leqslant C_1$ for every $1 \leqslant p < \frac{d}{d-1}$, where the constant C_1 depends neither on ε nor on η, but $C_1 \to +\infty$ if $\lambda \to 0$, $\delta \to 0$ or $p \to \frac{d}{d-1}$. Finally, given any $\psi \in C_c^\infty(\Omega)$, $\psi \geqslant 0$ in Ω, we insert $\varphi = \frac{\psi}{(\varepsilon+[h_{\varepsilon,\eta}]_\eta)^{1/3}}$ into (14) to get the inequality

$$\frac{2}{3}\int_\Omega \frac{\nabla h_{\varepsilon,\eta}\cdot\nabla\psi}{(\varepsilon+[h_{\varepsilon,\eta}]_\eta)^{1/3}} + \int_\Omega \frac{h_{\varepsilon,\eta}}{(\varepsilon+[h_{\varepsilon,\eta}]_\eta)^{1/3}}\psi \geqslant \int_\Omega \frac{|\nabla u_{\varepsilon,\eta}|^2}{1+\eta|\nabla u_{\varepsilon,\eta}|^2}\psi. \tag{16}$$

Passage to the limit $\eta \to 0$ ($\varepsilon > 0$ fixed). The estimates on $(u_{\varepsilon,\eta}, h_{\varepsilon,\eta})$ imply the existence of a subsequence (not relabelled) such that $(u_{\varepsilon,\eta}, h_{\varepsilon,\eta}) \to (u_\varepsilon, h_\varepsilon)$ weakly in $W_0^{1,2}(\Omega) \times W_{\Gamma_D}^{1,p}(\Omega)$ ($1 \leqslant p < \frac{d}{d-1}$) and a.e. in Ω as $\eta \to 0$. Clearly, $h_\varepsilon \geqslant 0$ a.e. in Ω. By a standard reasoning, $\eta \to 0$ in (15) yields

$$\int_\Omega \frac{|\nabla h_\varepsilon|^2}{(\lambda+h_\varepsilon)^{1+\delta}} \leqslant \frac{c}{\delta\lambda^\delta} \quad \forall\lambda>0,\ \forall\delta>0. \tag{17}$$

Now, the passage to the limit $\eta \to 0$ in (13) gives

$$\int_\Omega \big(\nu+(\varepsilon+h_\varepsilon)^{1/3}\big)\nabla u_\varepsilon\cdot\nabla\zeta = \int_\Omega f\zeta \quad \forall\zeta\in C_c^\infty(\Omega). \tag{18}$$

Taking $\lambda = \varepsilon$, $\delta = \frac{2}{3}$ in (17) we obtain $(\varepsilon+h_\varepsilon)^{1/6} \in W^{1,2}(\Omega)$. Then the approximation procedure in [3] gives the energy identity $\int_\Omega (\nu+(\varepsilon+h_\varepsilon)^{1/3})|\nabla u_\varepsilon|^2 = \int_\Omega f u_\varepsilon$. By the aid of this identity we can easily carry out the passage to the limit $\eta \to 0$ in (14), while $\eta \to 0$ in (16) is straightforward. Thus, (14) and (16) imply

$$\frac{2}{3}\int_\Omega \nabla h_\varepsilon\cdot\nabla\varphi + \int_\Omega h_\varepsilon\varphi = \int_\Omega (\varepsilon+h_\varepsilon)^{1/3}|\nabla u_\varepsilon|^2\varphi \quad \forall\varphi\in\bigcup_{s>d} W_{\Gamma_D}^{1,s}(\Omega), \tag{19}$$

$$-\int_\Omega (\varepsilon+h_\varepsilon)^{2/3}\Delta\psi + \int_\Omega \frac{h_\varepsilon\psi}{(\varepsilon+h_\varepsilon)^{1/3}} \geqslant \int_\Omega |\nabla u_\varepsilon|^2\psi \quad \forall\psi\in C_c^\infty(\Omega),\ \psi\geqslant 0 \text{ in } \Omega, \tag{20}$$

respectively.

Passage to the limit $\varepsilon \to 0$. It is readily seen that all estimates on $(u_{\varepsilon,\eta}, h_{\varepsilon,\eta})$ continue to hold for $(u_\varepsilon, h_\varepsilon)$ with the same constants. Again we can find a subsequence of $(u_\varepsilon, h_\varepsilon)$ (not relabelled) such that $(u_\varepsilon, h_\varepsilon) \to (u, h)$ with the same convergence properties as above for $(u_{\varepsilon,\eta}, h_{\varepsilon,\eta})$. Clearly, $h \geqslant 0$ a.e. in Ω. Then $\varepsilon \to 0$ in (18) and (20) gives the first integral identity in (8) and the inequality in (9), respectively (set $k := h^{2/3}$ a.e. in Ω).

It remains to carry out the passage to the limit $\varepsilon \to 0$ in (19) (thus proving (7)). As above, this is easily done with the help of the energy identity

$$\int_\Omega \big(\nu+h^{1/3}\big)|\nabla u|^2 = \int_\Omega f u. \tag{21}$$

To prove this identity, we notice that $\varepsilon \to 0$ in (17) gives

$$\int_\Omega \frac{|\nabla h|^2}{(\lambda+h)^{1+\delta}} \leqslant \frac{c}{\delta\lambda^\delta} \quad \forall\lambda>0,\ \forall\delta>0.$$

Hence, $(\lambda+h)^{(1-\delta)/2} \in W^{1,2}(\Omega)$ for all $0<\delta<1$. We take $\lambda = (\frac{\nu}{2})^3$, $\delta = \frac{2}{3}$, and define $\mu := \nu + h^{1/3} - (\lambda+h)^{1/3}$ a.e. in Ω. Then

$$\nu + h^{1/3} = \mu + (\lambda+h)^{1/3}, \quad \frac{\nu}{2} \leqslant \mu \leqslant \nu \text{ a.e. in } \Omega.$$

Now, the energy identity (21) can be proved by a slight modification of an approximation argument from [3].

Acknowledgement

The authors are deeply indebted to P. Penel for valuable remarks when preparing this Note.

References

[1] S. Clain, R. Touzani, Solution of a two-dimensional stationary induction heating problem without boundedness of the coefficients, Math. Model. Numer. Anal. 31 (1977) 845–870.
[2] P.-E. Druet, J. Naumann, On the existence of weak solutions to a stationary one-equation RANS model with unbounded eddy viscosities, Ann. Univ. Ferrara 55 (2009) 67–87.
[3] T. Gallouët, J. Lederer, R. Lewandowski, F. Murat, L. Tartar, On a turbulent system with unbounded viscosities, Nonlinear Anal. 52 (2003) 1051–1068.
[4] M. Jischa, Konvektiver Impuls-, Wärme- und Stoffaustausch, Vieweg-Verlag, Braunschweig/Wiesbaden, 1982.
[5] J. Lederer, R. Lewandowski, A RANS 3D model with unbounded eddy viscosities, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007) 413–441.
[6] J. Naumann, Existence of weak solutions to the equations of stationary motion of heat-conducting incompressible viscous fluids, in: Progress in Nonlinear Differential Equations and Their Applications, vol. 64, Birkhäuser-Verlag, 2005, pp. 373–390.
[7] S.B. Pope, Turbulent Flows, Cambridge Univ. Press, Cambridge, 2006.
[8] L. Prandtl, Über ein neues Formelsystem für die ausgebildete Turbulenz, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. (1945) 6–19.
[9] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier (Grenoble) 15 (1965) 189–258.
[10] E. Zeidler, Nonlinear Functional Analysis and Its Applications. II/B: Nonlinear Monotone Operators, Springer-Verlag, New York, 1990.