C. R. Acad. Sci. Paris, Ser. I 350 (2012) 67–70

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Differential Geometry

The Reeb foliation arises as a family of Legendrian submanifolds at the end of a deformation of the standard S^3 in S^5

Le feuilletage de Reeb se réalise comme une famille de sous-variétés legendriennes à l'aboutissement d'une déformation d'une sphère S^3 canonique dans S^5

Atsuhide Mori

Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

ARTICLE INFO

Article history:
Received 10 September 2010
Accepted after revision 3 January 2012
Available online 13 January 2012

Presented by Jean-Pierre Demailly

ABSTRACT

We realize the Reeb foliation of S^3 as a family of Legendrian submanifolds of the unit $S^5 \subset \mathbb{C}^3$. Moreover, we construct a deformation of the standard contact S^3 in S^5, via a family of contact submanifolds, into this realization.

RÉSUMÉ

Nous réalisons le feuilletage de Reeb comme une famille de sous-variétés legendriennes de la sphère unité S^5 dans $\mathbb{C}^3$. Par ailleurs, nous construisons une déformation de la structure de contact canonique S^3 dans S^5 via une famille de sous-variétés de contact, aboutissant au feuilletage ainsi réalisé.

1. Introduction

The Reeb foliation is a codimension one smooth foliation of the 3-sphere S^3 obtained by gluing two Reeb components $S^1 \times D^2$ and $D^2 \times S^1$. Since the one-sided holonomies of the Reeb components along $\{1\} \times \partial D^2$ and $\partial D^2 \times \{1\}$ are trivial, the Reeb foliation is not analytic ("Haefliger's remark").

On the other hand the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R}) \approx \mathbb{R}^{2n+1}$ for a function of n variables carries the canonical contact structure. It is contactomorphic to the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ minus any point. Here S^{2n+1} has the standard contact form $\alpha = \sum_{i=1}^{n+1} r_i^2\, \mathrm{d}\theta_i | S^{2n+1}$ ($r_i = |z_i|$, $\theta_i = \arg z_i$ for coordinates z_i of $\mathbb{C}^{n+1}$). Thus we may regard a codimension-n submanifold $M^{n+1} \subset S^{2n+1}$ as a system of n first-order partial differential equations (for implicit functions). If $\alpha \wedge \mathrm{d}\alpha | M^{n+1} = 0$ and $\alpha | M^{n+1} \neq 0$, the system is completely integrable and regular, and therefore defines a codimension one foliation $\mathcal{F}$ on M^{n+1}. The leaves of $\mathcal{F}$ are Legendrian submanifolds of S^{2n+1} corresponding to the solutions.

In this article we construct an embedding of S^3 into the standard S^5 so that the image has the Reeb foliation $\mathcal{F}$ by Legendrian submanifolds. This example shows that even a non-taut foliation can be a family of Legendrian submanifolds of $J^1(\mathbb{R}^n, \mathbb{R})$. Moreover we prove

E-mail address: ka-mori@ares.eonet.ne.jp.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2012.01.001

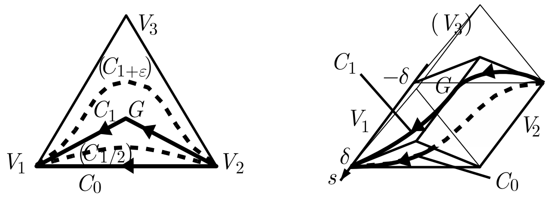

Fig. 1. The curve C_t on Δ and its parametrization by s.

Theorem 1.1. *There exists a smooth family* $\{M_t^3\}_{t\in[0,3/2)}$ *of codimension-2 submanifolds of* S^5 *such that*

(1) M_0^3 *is the standard* $S^3(\subset \mathbb{C}^2 \subset \mathbb{C}^3)$,
(2) M_t^3 *is an embedded contact submanifold for* $0 \leqslant t < 1$,
(3) M_1^3 *admits a Reeb foliation by injectively immersed Legendrian submanifolds of* S^5*, and*
(4) $-M_t^3$ *is an embedded overtwisted contact submanifold for* $1 < t < 3/2$.

The foliated submanifold M_1^3 is obtained by joining two great circles $\{r_1 = 1\}, \{r_2 = 1\} \subset S^5$ through the Legendrian torus $T = \{r_1 = r_2 = r_3 = 1/\sqrt{3},\ \theta_1 + \theta_2 + \theta_3 = 0\}$. The family M_t^3 is obtained as a byproduct in the process of isotoping $M_1 \subset S^5$ to the unknot. The author is seeking the converse approach, i.e., to find a foliated submanifold by using contact topology or open-books (see Remark 1 in Section 2).

2. Proof and remark

Proof. Let π be the natural projection of S^5 to the 2-simplex $\Delta = \{(r_1^2, r_2^2, r_3^2) \mid r_1^2 + r_2^2 + r_3^2 = 1\} \subset \mathbb{R}^3$, which sends the Legendrian 2-torus $T = \{r_1 = r_2 = r_3 = 1/\sqrt{3},\ \theta_1 + \theta_2 + \theta_3 = 0\} \subset S^5$ to the barycenter G. The set $\Gamma = \pi^{-1}(\partial\Delta)$ contains the great circles $\pi^{-1}(\{V_1, V_2, V_3\})$ where V_i denotes the vertex $r_i^2 = 1$. Except them $\pi|\Gamma$ is a T^2-fibration. On the other hand, $\pi|(S^5 \setminus \Gamma)$ is a T^3-fibration. Now we take the coordinates (x, y) on Δ by putting $\overrightarrow{OP} = \overrightarrow{OG} + x\overrightarrow{GV_1} + y\overrightarrow{GV_2}$ for $P \in \Delta$, i.e.,

$$3r_1^2 = 1 + 2x - y(\geqslant 0), \qquad 3r_2^2 = 1 - x + 2y(\geqslant 0), \quad \text{and} \quad 3r_3^2 = 1 - x - y(\geqslant 0).$$

Let M_0^3 be the standard $S^3 = \pi^{-1}(\overline{V_1V_2})$. We deform M_0^3 with the help of a certain family of simple curves C_t: $x = x_t(s)$, $y = y_t(s)$, $-\delta \leqslant s \leqslant \delta$ depicted in Fig. 1 ($0 < \delta \ll 1$, $0 \leqslant t \leqslant 3/2$). Note that C_1 has a break point G while $x_1(s)$ and $y_1(s)$ are smooth on $(-\delta, \delta)$.

We generate $M_t^3 \subset S^5$ by moving the intersection of the "wall" $W_s = \mathrm{cl}\{\theta_1 + \theta_2 + \theta_3 = s\} \subset S^5$ with the fiber $\pi^{-1}(x_t(s), y_t(s))$ for $-\delta \leqslant s \leqslant \delta$. Then we can see that M_t^3 realizes the join of two large circles $\pi^{-1}(V_2)$ and $\pi^{-1}(V_1)$. Now we give a precise definition of the curve C_t. Put $\varphi_0(u) = \frac{1}{2}(1 + u)$ for $u \in [-1, 1]$, and take a smooth function $\varphi_1(u)$ and a smooth odd function $s(u)$ such that

$$\varphi_1(u) = 0 \quad (-1 \leqslant u \leqslant 0), \qquad \varphi_1'(u) > 0 \quad (0 < u \leqslant 1), \qquad \varphi_1(u) = \varphi_0(u) \quad (1/2 \leqslant u \leqslant 1),$$
$$s'(u) > 0 \quad (-1 < u < 1), \qquad s(1) = \delta, \qquad s(-1) = -\delta, \quad \text{and} \quad s(u) \text{ is } C^\infty\text{-tangent to } \pm\delta.$$

The inverse function $u(s)$ of $s(u)$ is defined on $[-\delta, \delta]$. It is smooth on $(-\delta, \delta)$ ($u'(\pm\delta) = +\infty$). We put $\varphi_t(u) = (1-t)\varphi_0(u) + t\varphi_1(u)$, and take the curve

$$C_t\colon \quad x = x_t(s) = \varphi_t\big(u(s)\big), \quad y = y_t(s) = \varphi_t\big(u(-s)\big), \quad -\delta \leqslant s \leqslant \delta.$$

Next we show that M_t^3 is a smooth submanifold. By moving the 2-torus $(M_t^3 \setminus \Gamma) \cap W_s$ for $-\delta < s < \delta$, we see that $M_t^3 \setminus \Gamma$ is diffeomorphic to $T^2 \times (-\delta, \delta)$. Moreover M_t^3 is topologically the join $S^1 \star S^1 \approx S^3$. Thus it only remains for us to examine the smoothness of M_t^3 along $M_t^3 \cap \Gamma$. We restrict ourself to the connected component of $M_t^3 \cap \Gamma$ corresponding to $s = +\delta$ and omit the other component. We put

$$\widetilde{M_t^3}: \begin{cases} r_1^2 + r_2^2 + r_3^2 = 1, \\ 3r_2^2 = 1 - \dfrac{1}{2}(1+u) + (1-t)(1-u), \\ 3r_3^2 = 1 - \dfrac{1}{2}(1+u) - \dfrac{1-t}{2}(1-u) = \dfrac{t}{3-2t} \cdot 3r_2^2, \\ \theta_1 + \theta_2 + \theta_3 = 1 \end{cases}$$

where $u \in [1/2, 1]$ is a parameter to be eliminated. Then $\{\theta_1 = \text{const}\} \subset \widetilde{M_t^3}$ is a smooth disk since it tangents to the real 2-plane $\{z_1 = \exp\sqrt{-1}\theta_1,\ z_3 = \overline{z_2} \cdot \sqrt{\frac{t}{3-2t}} \exp\{\sqrt{-1}(1-\theta_1)\}\} \subset \mathbb{C}^3$ at $u = 1$. Since the function $s(u)$ smoothly tangents to δ at $u = 1$, M_t^3 is a smooth 3-sphere.

Next we consider the (non-)integrability of the restriction $\lambda_t = \alpha|M_t^3$ of the standard contact form $\alpha = r_1^2\,\mathrm{d}\theta_1 + r_2^2\,\mathrm{d}\theta_2 + r_3^2\,\mathrm{d}\theta_3|S^5$. Using (θ_1, θ_2, s) as coordinates of $M_t^3 \setminus \Gamma$, we can write

$$\lambda_t = x_t(s)\,\mathrm{d}\theta_1 + y_t(s)\,\mathrm{d}\theta_2 + \bigl(1 - x_t(s) - y_t(s)\bigr)\,\mathrm{d}s.$$

Here the sign of $\lambda_t \wedge d\lambda_t$ with respect to $\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \wedge \mathrm{d}s > 0$ coincides with that of $x_t'(s)y_t(s) - x_t(s)y_t'(s)$, and that of $1 - t$. More generally, if a submanifold $M^3 (\approx T^2 \times \mathbb{R}) \subset S^5$ is presented by a simple curve C: $x = x(s)$, $y = y(s)$ on $\operatorname{int}\Delta$, the negative areal velocity $x'(s)y(s) - x(s)y'(s)$ still presents the non-integrability of $\alpha|M^3$. In the case where $t = 1$, the integrability means the vanishing of the areal velocity. That is why the curve C_1 is broken into two rays to/from the origin G, and M_1^3 is non-analytic.

On the other hand, for cylindrical coordinates $(\theta_1, (r_2, \theta_2))$, $\mu_t = \alpha|\widetilde{M_t^3}$ and $\mu_t \wedge \mathrm{d}\mu_t$ are written as

$$\mu_t = \left(1 - \frac{3}{3-2t} r_2^2\right)\mathrm{d}\theta_1 + \frac{3(1-t)}{3-2t} r_2^2\,\mathrm{d}\theta_2 \quad \text{and} \quad \mu_t \wedge \mathrm{d}\mu_t = \frac{6(1-t)}{3-2t}\,\mathrm{d}\theta_1 \wedge (r_2\,\mathrm{d}r_2 \wedge \mathrm{d}\theta_2).$$

This implies that the sign of $\lambda_t \wedge \mathrm{d}\lambda_t$ everywhere coincides with that of $1 - t$.

Now we show that the foliation of M_1^3 is a Reeb foliation. The definition of M_1^3 is

$$\begin{cases} 3r_1^2 = 1 + 2\varphi_1\bigl(u(s)\bigr) - \varphi_1\bigl(u(-s)\bigr), \\ 3r_2^2 = 1 - \varphi_1\bigl(u(s)\bigr) + 2\varphi_1\bigl(u(-s)\bigr), \\ 3r_3^2 = 1 - \varphi_1\bigl(u(s)\bigr) - \varphi_1\bigl(u(-s)\bigr), \\ \theta_1 + \theta_2 + \theta_3 = s \end{cases}$$

where $s \in [-\delta, \delta]$ is a parameter to be eliminated. On the open solid torus $H = \{s > 0\} \subset M_1^3$, we have

$$\alpha|H = \varphi_1\bigl(u(s)\bigr)\,\mathrm{d}\theta_1 + \bigl\{1 - \varphi_1\bigl(u(s)\bigr)\bigr\}\,\mathrm{d}s.$$

Thus the surface of θ_2-revolution of the graph of $\theta_1 = \int \frac{\varphi_1(u(s))-1}{\varphi_1(u(s))}\,\mathrm{d}s$ is a leaf. Similarly, we can describe the foliation on $\{s < 0\}$. These foliations spiral into T and form a transversely oriented Reeb foliation, to which the positive Hopf link $\{r_1 = 1\} \cup \{r_2 = 1\}$ is positively transverse.

Finally we see from $\mathrm{d}(\theta_1 + \theta_2) \wedge \mathrm{d}\lambda_t = \{x_t'(s) - y_t'(s)\}\,\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \wedge \mathrm{d}s > 0$ $(t \neq 1)$ that the positive Hopf band $\ker(\mathrm{d}\theta_1 + \mathrm{d}\theta_2)$ is a supporting open-book for $0 \leqslant t < 1$. On the other hand, the negative Hopf band $\ker(-\mathrm{d}\theta_1 - \mathrm{d}\theta_2)$ on $-M_t (\approx S^3)$ is a supporting open-book for $1 < t < 3/2$. Thus $-M_t^3$ is overtwisted. Indeed it has the half-Lutz tube $\{x_t(s) \leqslant 0\}$. Moreover, since we can reverse the orientation of S^3 by a diffeotopy, we obtain the following "negative stabilization" lemma. This ends the proof.

Lemma 2.1. *The overtwisted contact submanifold $-M_{5/4}^3 \subset S^5$ is diffeotopic to the standard $S^3 \subset S^5$. Particularly $-M_{5/4}^3$ is differential topologically unknotted, but contact topologically knotted.*

Remark 1. Any closed oriented 3-manifold admits an open-book decomposition (Alexander [1]). We can associate to it a contact structure (Thurston and Winkelnkemper [8]) as well as a spinnable foliation (see [5]). Further any contact structure is supported by an open-book decomposition (Giroux [3]). Using this result, the author constructed a certain immersion of any contact 3-manifold into $J^1(\mathbb{R}^2, \mathbb{R})$ or S^5 [6]. This construction was generalized to any dimension, i.e., $M^{2n+1} \to J^1(\mathbb{R}^{2n}, \mathbb{R})$ or S^{4n+1} by Martínez Torres [4]. The author proved that any/some contact structure of M^3 can be deformed into some/any spinnable foliation ([5], see also [2]). He also proved that a certain higher dimensional contact structure can be deformed into a foliation [7]. It is interesting to generalize the present result to these cases.

Acknowledgements

The author would like to thank the anonymous reviewer(s) for encouragement to include intuitive descriptions and a figure, which have made the article easier to access for general readers.

References

[1] J. Alexander, A lemma on systems of knotted curves, Proc. Nat. Acad. Sci. USA 9 (1923) 93–95.
[2] J. Etnyre, Contact structures on 3-manifolds are deformations of foliations, Int. Math. Res. Notices 14 (2007) 775–779.
[3] E. Giroux, Géometrie de contact de la dimension trois vers les dimensions supérieures, in: Proc. ICM2002, Beijing, vol. II, pp. 405–414.

[4] D. Martínez Torres, Contact embeddings in standard contact spheres via approximately holomorphic geometry, J. Math. Sci. Univ. Tokyo 18 (2011) 139–154.
[5] A. Mori, A note on Thurston–Winkelnkemper's construction of contact forms on 3-manifolds, Osaka J. Math. 39 (2002) 1–11.
[6] A. Mori, Global models of contact forms, J. Math. Sci. Univ. Tokyo 11 (2004) 447–454.
[7] A. Mori, Reeb foliations on S^5 and contact 5-manifolds violating the Thurston–Bennequin inequality, preprint, 2009, arXiv:0906.3237.
[8] W. Thurston, H. Winkelnkemper, On the existence of contact forms, Proc. AMS 52 (1975) 345–347.