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Complex Analysis

A remark on the Bergman kernels of the Cartan–Hartogs domains

Une remarque sur le noyau de Bergman des domaines de Cartan–Hartogs

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ABSTRACT

We give a new formula for the Bergman kernels of the Cartan–Hartogs domains. As an application of our formula, we study the Lu Qi-Keng problem of the Cartan–Hartogs domains.

RÉSUMÉ

Nous obtenons une nouvelle formule pour le noyau de Bergman des domaines de Cartan–Hartogs. Comme application, nous étudions le problème du Lu QiKeng pour les domaines de Cartan–Hartogs.

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1. Introduction and preliminaries

Let R be a Cartan domain and N its generic norm. The focus of this paper is on the Bergman kernel of the Cartan–Hartogs domain $\Omega_R := \{(z,\zeta) \in R \times \mathbb{C}^m;\ \|\zeta\|^2 < N(z,z)^s\}$ where $s > 0$. The domain Ω_R was introduced by W. Yin and G. Roos and an explicit formula of the Bergman kernel was given in [6]. One of the main result of this Note is to give another expression of the Bergman kernel in terms of the polylogarithm function. Our approach for the Bergman kernel is based on the Forelli–Rudin construction, which was proved by E. Ligocka [4]. It is a series representation formula of the Bergman kernel of the Hartogs domain involving weighted Bergman kernels of the base domain.

As an application of our formula, we study the Lu Qi-Keng problem of the Cartan–Hartogs domain, which asks whether or not the Bergman kernel is zero-free. The domain is called a Lu Qi-Keng domain if its Bergman kernel is zero-free. For the motivation of the Lu Qi-Keng problem, see [1].

There are some results on the Lu Qi-Keng problem of the Cartan–Hartogs domain. The Lu Qi-Keng problem of the Cartan–Hartogs domain was completely solved in [3] when the base domain is the Cartan domain of dimension less or equal to 4. Define $\Omega^{s,1}_{n,m} := \{(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^m;\ \|\zeta\|^{2s} + \|z\|^2 < 1\}$, which is a special case of the Cartan–Hartogs domain. L. Zhang and W. Yin [9, Theorem 1(1)] proved the following result:

Theorem 1.1. *For fixed n and s, there exists a constant $m_0 = m_0(n,s)$ such that the domain $\Omega^{s,1}_{n,m}$ is a Lu Qi-Keng domain for all $m \geqslant m_0$.*

It is natural to expect that an analogous result of Theorem 1.1 also holds for the Cartan–Hartogs domain. Our second result is to show that this expectation is true (Theorem 3.3).

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1.1. Hua polynomial and weighted Bergman kernel

Let R be a Cartan domain. We denote by a, b, and r, the characteristic multiplicities and the rank of R respectively. The Hua polynomial $\chi(s)$ of R is defined by the Hua integral:

$$\int_R N(z,z)^s \,\mathrm{d}V(z) = \frac{\chi(0)}{\chi(s)} \int_R \mathrm{d}V(z), \quad s > 0.$$

It is known that $\chi(s) = \prod_{j=1}^{r} (s + 1 + (j-1)\frac{a}{2})_{1+b+(r-j)a}$, where $(x)_k = x(x+1)\cdots(x+k-1)$. From this expression, it is easy to see that all coefficients of $\chi(s)$ are positive. It is known that the Hua polynomial appears in the reproducing kernel K_{R,N^s} of the weighted Bergman space $L_a^2(R, N(z,z)^s)$ (see [5, Corollary 2.2] and [3, Lemme 1]):

$$K_{R,N^s}\big(z,z'\big) = \frac{\chi(s)}{\chi(0)} N\big(z,z'\big)^{-s} K_R\big(z,z'\big), \tag{1}$$

where K_R is the (unweighted) Bergman kernel of R. We will see that Eq. (1) plays an important role in the proof of our theorem.

1.2. Polylogarithm function

We define the polylogarithm function by $Li_s(t) := \sum_{k=1}^{\infty} k^{-s} t^k$, which converges under $|t| < 1$, $s \in \mathbb{C}$. From the definition of $Li_s(t)$, we have the series representation of the m-th derivative of the polylogarithm:

$$\frac{\mathrm{d}^m Li_{-n}(t)}{\mathrm{d}t^m} = \sum_{k=0}^{\infty} (k+1)_m (k+m)^n t^k. \tag{2}$$

If s is a negative integer, say $s = -n$, then the polylogarithm has the following expression [2, eq. 2.10c]:

$$Li_{-n}(t) = \sum_{j=0}^{n} \frac{(-1)^{n+j} j! S(1+n, 1+j)}{(1-t)^{j+1}}, \tag{3}$$

where $S(\cdot,\cdot)$ denotes the Stirling number of the second kind. As a consequence we have the following:

$$\frac{\mathrm{d}^m Li_{-n}(t)}{\mathrm{d}t^m} = \frac{m! \sum_{j=0}^{n} (-1)^{n+j} (m+1)_j S(1+n, 1+j)(1-t)^{n-j}}{(1-t)^{n+m+1}}. \tag{4}$$

2. Bergman kernel

In our preprint [8], we gave an explicit expression of the Bergman kernel of the domain $\Omega_{n,m}^{s,1}$ in terms of the polylogarithm function. Now we generalize our previous result for the Cartan–Hartogs domain.

Theorem 2.1. *The Bergman kernel K_{Ω_R} of the Cartan–Hartogs domain Ω_R is given by*

$$K_{\Omega_R}\big((z,\zeta),\big(z',\zeta'\big)\big) = \frac{K_R(z,z')}{\pi^m \chi(0) N(z,z')^{sm}} \sum_{\ell=0}^{d} s^\ell c_\ell \frac{\mathrm{d}^m}{\mathrm{d}t^m} Li_{-\ell}(t)\Big|_{t=N(z,z')^{-s}\langle\zeta,\zeta'\rangle}, \tag{5}$$

where we put $\deg \chi = d$ *and* $\chi(s) = \sum_{\ell=0}^{d} c_\ell s^\ell$.

Proof. By Ligocka's formula [4, Proposition 0] (see also [5, Theorem 1.2]), we have

$$K_{\Omega_R}\big((z,\zeta),\big(z',\zeta'\big)\big) = \sum_{k=0}^{\infty} \frac{(k+1)_m}{\pi^m} K_{R,N^{s(k+m)}}\big(z,z'\big)\big\langle\zeta,\zeta'\big\rangle^k. \tag{6}$$

Using (1), we know that the right-hand side of (6) is equal to

$$\frac{K_R(z,z')}{\pi^m \chi(0) N(z,z')^{sm}} \sum_{k=0}^{\infty} (k+1)_m \chi\big(s(k+m)\big) t^k,$$

where $t = N(z,z')^{-s}\langle\zeta,\zeta'\rangle$. We easily see from (2) that

$$\sum_{k=0}^{\infty}(k+1)_m\chi\big(s(k+m)\big)t^k=\sum_{\ell=0}^{d}s^\ell c_\ell\sum_{k=0}^{\infty}(k+1)_m(k+m)^\ell t^k=\sum_{\ell=0}^{d}s^\ell c_\ell\frac{\mathrm{d}^m}{\mathrm{d}t^m}Li_{-\ell}(t).$$

We have thus proved the theorem. □

Remark 1. In a completely analogous way, we can obtain an explicit formula of the Bergman kernel of the domain $D_{n,m}:=\{(z,\zeta)\in\mathbb{C}^n\times\mathbb{C}^m;\ \|\zeta\|^2<e^{-s\|z\|^2}\}$ (see [7]).

3. Zeros of the Bergman kernel

As an application of our result, we study the Lu Qi-Keng problem of the Cartan–Hartogs domain. We begin the study with the following lemma. Since the proof of the lemma is straightforward, we omit it.

Lemma 3.1. *For any $(z,\zeta),(z',\zeta')\in\Omega_R\times\Omega_R$, we have $|N(z,z')^{-s}\langle\zeta,\zeta'\rangle|<1$.*

It is well known that the Bergman kernel of the Cartan domain is zero-free. Using this fact, Theorem 2.1 and Lemma 3.1, we know the following proposition:

Proposition 3.2. *Put $F(t)=\sum_{\ell=0}^{d}s^\ell c_\ell\frac{\mathrm{d}^m}{\mathrm{d}t^m}Li_{-\ell}(t)$. The Cartan–Hartogs domain is a Lu Qi-Keng domain if all roots of $F(t)$ lie outside the unit circle.*

Now we state our result, which is a generalization of Theorem 1.1:

Theorem 3.3. *There exists a number $m_0\in\mathbb{N}$ such that the Cartan–Hartogs domain Ω_R is a Lu Qi-Keng domain for any $m\geqslant m_0$. The number m_0 depends on the constant s and the base domain R.*

Proof. Define the polynomial $A_{n,m}(t)$ by

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}Li_{-n}(t)=\frac{m!A_{n,m}(t)}{(1-t)^{1+n+m}}.$$

Then we have

$$\sum_{\ell=0}^{d}s^\ell c_\ell\frac{\mathrm{d}^m}{\mathrm{d}t^m}Li_{-\ell}(t)=\frac{m!\sum_{k=0}^{d}s^k c_k(1-t)^{d-k}A_{k,m}(t)}{(1-t)^{m+d+1}}.$$

Using (4), we decompose the polynomial $G(t)=\sum_{k=0}^{d}s^k c_k(1-t)^{d-k}A_{k,m}(t)$ as follows:

$$G(t)=G_1(t)+G_2(t)+s^d c_d(m+1)_d,$$

where we put

$$G_1(t)=s^d c_d\sum_{j=0}^{d-1}(-1)^{d+j}(m+1)_j S(1+k,1+j)(1-t)^{d-j},$$

$$G_2(t)=\sum_{k=0}^{d-1}s^k c_k\sum_{j=0}^{k}(-1)^{k+j}(m+1)_j S(1+k,1+j)(1-t)^{d-j}.$$

Then it is easy to see that

$$\big|G_1(t)+G_2(t)\big|<s^d c_d\sum_{j=0}^{d-1}(m+1)_j S(1+d,1+j)2^d+\sum_{k=0}^{d-1}s^k c_k\sum_{j=0}^{k}(m+1)_j S(1+k,1+j)2^d,\tag{7}$$

on $|t|=1$. We regard the right-hand side of (7) as a polynomial $H(m)$ in m. The degree of $H(m)$ is $d-1$ and its leading coefficient is positive. On the other hand, the degree of the polynomial $s^d c_d(m+1)_d$ is d as a polynomial in m. Thus there exists a number m_0 such that $|G_1(t)+G_2(t)|<H(m)<s^d c_d(m+1)_d$ for $m\geqslant m_0$ on $|t|=1$. Hence, due to Rouché's theorem, we conclude that all roots of $G(t)$ lie outside the unit circle for $m\geqslant m_0$. We have thus proved the theorem. □

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