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Algebraic Geometry

On vector bundles on curves over $\overline{\mathbb{F}}_p$

Sur les fibrés vectoriels sur les courbes sur le corps $\overline{\mathbb{F}}_p$

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ABSTRACT

Let V be a vector bundle over an irreducible smooth projective curve defined over the field $\overline{\mathbb{F}}_p$. For any integer $r \in (0, \operatorname{rank}(V))$, let $\operatorname{Gr}_r(V)$ be the Grassmann bundle parametrizing r-dimensional quotients of the fibers of V. Let L be a line bundle over $\operatorname{Gr}_r(V)$ such that $L \cdot C > 0$ for every irreducible closed curve $C \subset \operatorname{Gr}_r(V)$. We prove that L is ample.

RÉSUMÉ

Soit V un fibré vectoriel sur une courbe projective lisse irréductible définie sur $\overline{\mathbb{F}}_p$. Pour tout entier $r \in (0, \operatorname{rank}(V))$, soit $\operatorname{Gr}_r(V)$ le fibré en grassmanniennes paramétrisant les quotients de dimension r des fibrés de V. Soit L un fibré en droites sur $\operatorname{Gr}_r(V)$ tel que $L \cdot C > 0$ pour toute courbe fermée irréductible $C \subset \operatorname{Gr}_r(V)$. On prouve alors que L est ample.

1. Introduction

There are many examples of non-ample line bundles, over some projective variety defined over an algebraically closed field, whose restriction to every closed curve is of positive degree. Somehow, for all known examples, the base is a projective bundle over a curve (see [6, p. 56, Example 10.6] (this example is due to Mumford), [10,8]). On the other hand, a question due to S. Keel asks whether a line bundle L over a smooth projective surface Z defined over $\overline{\mathbb{F}}_p$, with the property that the restriction of L to every closed curve in Z is of positive degree, is ample (see [7, p. 3959, Question 0.9]). In [4], this question was answered affirmatively for $\mathbb{P}^1$-bundles over curves.

Let M be an irreducible smooth projective curve defined over $\overline{\mathbb{F}}_p$. Let V be a vector bundle over M. For any integer $r \in (0, \operatorname{rank}(V))$, let $\operatorname{Gr}_r(V)$ be the Grassmann bundle over M parametrizing r-dimensional quotients of the fibers of V.

Our aim here is to prove the following (see Theorem 2.2):

Theorem 1.1. *Let L be a line bundle over $\operatorname{Gr}_r(V)$ such that*

$$L \cdot C > 0$$

for every irreducible closed curve $C \subset \operatorname{Gr}_r(V)$. Then L is ample.

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2. Splitting over some pullback

Let p be a prime number. The field with p elements will be denoted by $\mathbb{F}_p$. Let X be a smooth irreducible projective curve defined over $\overline{\mathbb{F}}_p$. Let

$$F_X : X \to X$$

be the absolute Frobenius morphism. For any positive integer n, let

$$F_X^n := \overbrace{F_X \circ \cdots \circ F_X}^{n\text{-times}} : X \to X$$

be the n-fold iteration of F_X.

Proposition 2.1. *Let V be a vector bundle on X of rank r defined over $\overline{\mathbb{F}}_p$. There is an irreducible smooth projective curve Y, a non-constant morphism*

$$f : Y \to X$$

and a line bundle $L \to Y$, such that

$$f^*V = \bigoplus_{i=1}^{r} L^{\otimes a_i},$$

where $a_i \in \mathbb{Z}$.

Proof. There is a positive integer n such that

$$\left(F_X^n\right)^* V = \bigoplus_{i=1}^{\ell} W_i$$

with each W_i strongly semistable [3, p. 356, Proposition 2.1]. Let $c \in \mathbb{N}^+$ be a common multiple of the integers $\operatorname{rank}(W_i)$, $1 \leqslant i \leqslant \ell$. There is an irreducible smooth projective curve Z defined over $\overline{\mathbb{F}}_p$ and a non-constant morphism

$$h : Z \to X$$

of degree c. Indeed, if p^b is the largest power of p in the factorization of c, then by using F_X^b we may replace c by c/p^b, meaning if $h : Z \to X$ is of degree c/p^b, then $F_X^b \circ h$ is of degree c. Now we note that there are separable (possibly ramified) coverings of X of degree c/p^b.

Consider, the pullback

$$h^*\left(F_X^n\right)^* V = \bigoplus_{i=1}^{\ell} h^* W_i.$$

Let d_i be the degree of h^*W_i. Note that d_i is a multiple of $\operatorname{rank}(W_i)$ for each i, because d_i is a multiple of c. Define

$$e_i := -\frac{d_i}{\operatorname{rank}(W_i)} \in \mathbb{Z}.$$

Fix a line bundle ξ on Z of degree one. Take any $i \in [1, \ell]$. Consider the vector bundle $h^*W_i \otimes \xi^{\otimes e_i}$, where e_i is defined above. Note that it is strongly semistable of degree zero. Therefore, $h^*W_i \otimes \xi^{\otimes e_i}$ is essentially finite [11], [1, Theorem 1.1] (here it is used that Z is defined over $\overline{\mathbb{F}}_p$); see [9] for the definition of essentially finite vector bundles. Since $h^*W_i \otimes \xi^{\otimes e_i}$ admits a reduction of structure group to a finite group-scheme (as it is essentially finite), there is an irreducible smooth projective curve Z_i and a non-constant morphism

$$f_i : Z_i \to Z$$

such that the vector bundle $f_i^*(h^*W_i \otimes \xi^{\otimes e_i})$ is trivial (see [2, p. 557]). Consequently,

$$f_i^* H^* W_i = \mathcal{O}_{Z_i}^{\oplus r_i} \otimes f_i^* \xi^{\otimes -e_i},$$

where r_i is the rank of W_i.

Consider the fiber product

$$Z_1 \times_Z Z_2 \times_Z \cdots \times_Z Z_\ell \to Z.$$

Let Z' be an irreducible subscheme of it of dimension one. Let Y be the normalization of the reduced scheme Z'_{red}. Let f be the natural projection of Y to X. Take L to be the pullback of the line bundle $\xi \to Z$ to Y by the obvious morphism $Y \to Z$. This triple (Y, f, L) evidently satisfies the condition in the proposition. □

Let V be a vector bundle on an irreducible smooth projective curve M defined over $\overline{\mathbb{F}}_p$. Take any integer $r \in (0, \mathrm{rank}(V))$. Let $\mathrm{Gr}_r(V)$ be the Grassmann bundle over M parametrizing the quotient linear spaces of the fibers of V of dimension r.

Theorem 2.2. *Let L be a line bundle over $\mathrm{Gr}_r(V)$ such that for each irreducible closed curve $C \subset \mathrm{Gr}_r(V)$, we have*

$$L \cdot C > 0.$$

Then L is ample.

Proof. Let $f : \mathrm{Gr}_r(V) \to M$ be the natural projection. The tautological line bundle on $\mathrm{Gr}_r(V)$ will be denoted by $\mathcal{O}_{\mathrm{Gr}_r(V)}(1)$. Let d be the positive integer such that $L|_{f^{-1}(x)} = \mathcal{O}_{\mathrm{Gr}_r(V)}(d)|_{f^{-1}(x)}$ for any $x \in M$ (recall that the Picard group of a Grassmannian is generated by the tautological line bundle). Then by the see-saw principle, there is a unique line bundle ξ on M such that

$$L = \mathcal{O}_{\mathrm{Gr}_r(V)}(d) \otimes f^*\xi. \tag{1}$$

Take a pair (X, ϕ), where X is an irreducible smooth projective curve defined over $\overline{\mathbb{F}}_p$, and

$$\phi : X \to M$$

is a non-constant morphism of degree $d \cdot r$; we saw in the proof of Proposition 2.1 that such a pair exists. So degree$(\phi^*\xi)$ is a multiple of $d \cdot r$. Fix a line bundle ξ_0 on X such that

$$\xi_0^{\otimes dr} = \phi^*\xi. \tag{2}$$

Define the vector bundle

$$E := \big(\phi^* V\big) \otimes \xi_0.$$

Let $\mathrm{Gr}_r(E) \to X$ be the Grassmann bundle parametrizing the quotients of fibers of E of dimension r. The tautological line bundle on $\mathrm{Gr}_r(E)$ will be denoted by $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$. Since $\mathrm{Gr}_r(E) = \mathrm{Gr}_r(\phi^* V)$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_r(E) = \mathrm{Gr}_r(\phi^* V) & \xrightarrow{\beta} & \mathrm{Gr}_r(V) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\phi} & M \end{array} \tag{3}$$

From the construction of E, and the isomorphisms in (1) and (2), it follows immediately that

$$\beta^* L = \mathcal{O}_{\mathrm{Gr}_r(E)}(d). \tag{4}$$

Since the above morphism β is finite surjective flat, and $d > 0$, from (4) it follows that the line bundle L is ample if $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is ample [5, p. 73, Proposition 4.3]. Note that $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is nef because L is so.

In view of Proposition 2.1, we may assume that there is a line bundle $\mathcal{L}$ on X such that

$$E = \bigoplus_{i=1}^{\mathrm{rank}(E)} \mathcal{L}^{a_i} \tag{5}$$

(all we need to do is to replace X by another suitable smooth irreducible projective curve X' mapping to X, and replace ϕ by the composition $X' \to M$).

Let $\mathbb{P}(\bigwedge^r E) \to X$ be the projective bundle. Let $\mathcal{O}_{\mathbb{P}(\bigwedge^r E)}(1)$ be the tautological line bundle over $\mathbb{P}(\bigwedge^r E)$. We have the Plücker embedding

$$\varphi : \mathrm{Gr}_r(E) \to \mathbb{P}\Big(\bigwedge^r E\Big),$$

for which there is natural isomorphism

$$\varphi^* \mathcal{O}_{\mathbb{P}(\bigwedge^r E)}(1) \xrightarrow{\sim} \mathcal{O}_{\mathrm{Gr}_r(E)}(1). \tag{6}$$

As noted above, to prove the theorem it suffices to show that $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is ample. In view of (6), the line bundle $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is ample if the line bundle $\mathcal{O}_{\mathbb{P}(\bigwedge^r E)}(1)$ on $\mathbb{P}(\bigwedge^r E)$ is ample. Therefore, it is enough to show that the vector bundle $\bigwedge^r E$ is ample.

From (5) we conclude that

$$\bigwedge^r E = \bigoplus_{1\leqslant i_1<i_2<\cdots<i_r\leqslant \mathrm{rank}(E)} \mathcal{L}^{\otimes \sum_{j=1}^r a_{i_j}}. \tag{7}$$

Since a direct sum of ample line bundles is ample, from (7) it follows that $\bigwedge^r E$ is ample if

$$\mathrm{degree}\big(\mathcal{L}^{\otimes \sum_{j=1}^r a_{i_j}}\big) = \left(\sum_{j=1}^r a_{i_j}\right) \cdot \mathrm{degree}(\mathcal{L}) > 0 \tag{8}$$

for all $\{i_j\}_{j=1}^r$ such that $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant \mathrm{rank}(E)$.

Since the morphism β in (3) is finite, the given condition that $L \cdot C > 0$ for all $C \subset \mathrm{Gr}_r(V)$, and (4) together imply that

$$\mathcal{O}_{\mathrm{Gr}_r(E)}(1) \cdot D > 0 \tag{9}$$

for every irreducible closed curve $D \subset \mathrm{Gr}_r(E)$. Taking D to be the image of the section $X \to \mathrm{Gr}_r(E)$ corresponding to the quotient

$$E \to \bigoplus_{j=1}^r \mathcal{L}^{a_{i_j}},$$

where $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant \mathrm{rank}(E)$, from (9) we conclude that (8) holds. This completes the proof of the theorem. □

References

[1] I. Biswas, On principal bundles over a projective variety defined over a finite field, J. K-Theory 4 (2009) 209–221.
[2] I. Biswas, Y.I. Holla, Comparison of fundamental group schemes of a projective variety and an ample hypersurface, J. Algebraic Geom. 16 (2007) 547–597.
[3] I. Biswas, A.J. Parameswaran, On the ample vector bundles over curves in positive characteristic, C. R. Acad. Sci. Paris, Ser. I 339 (2004) 355–358.
[4] I. Biswas, S. Subramanian, On a question of Sean Keel, J. Pure Appl. Algebra 215 (2011) 2600–2602.
[5] R. Hartshorne, Ample vector bundles, Inst. Hautes Études Sci. Publ. Math. 29 (1966) 63–94.
[6] R. Hartshorne, Ample Subvarieties of Algebraic Varieties, Lecture Notes in Math., vol. 156, Springer-Verlag, Berlin–Heidelberg–New York, 1970.
[7] S. Keel, Polarized pushouts over finite fields, Comm. Algebra 31 (2003) 3955–3982.
[8] V. Mehta, S. Subramanian, Nef line bundles which are not ample, Math. Z. 219 (1995) 235–244.
[9] M.V. Nori, The fundamental group-scheme, Proc. Indian Acad. Sci. Math. Sci. 91 (1982) 73–122.
[10] S. Subramanian, Mumford's example and a general construction, Proc. Indian Acad. Sci. Math. Sci. 99 (1989) 197–208.
[11] S. Subramanian, Strongly semistable bundles on a curve over a finite field, Arch. Math. 89 (2007) 68–72.