



Complex Analysis/Functional Analysis

Some properties of composition operators on entire Dirichlet series with real frequencies

Quelques propriétés des opérateurs de composition sur les séries de Dirichlet entières à fréquences réelles

Hou Xiaolu, Le Hai Khoi

Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University (NTU), 637371 Singapore

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ABSTRACT

In this Note we consider some problems for composition operators on a class of entire Dirichlet series with real frequencies in the complex plane whose Ritt order is zero and logarithmic orders are finite. Criteria for action and boundedness of such operators are given.

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R É S U M É

Dans cette Note nous considérons quelques problèmes concernant les opérateurs de composition sur une classe de séries de Dirichlet entières à fréquences réelles dans le plan complexe, dont l'ordre de Ritt est zéro, et dont les ordres logarithmiques sont finis. Nous donnons des critères d'action et de continuité pour de tels opérateurs.

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1. Introduction

1.1. Basic notation and definitions

Let $0 \leq (\lambda_n)_{n=1}^\infty \uparrow \infty$ be given. General Dirichlet series with real frequencies $\Lambda = (\lambda_n)$ are the series of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \quad a_n, z \in \mathbb{C}. \quad (1)$$

When $\lambda_n = \log n$, series (1) become classical Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ which have important applications in number theory.

The abscissa of convergence σ_c , absolute convergence σ_a , or uniform convergence σ_u of series (1) is defined as the greatest lower bound of those values of $r \in \mathbb{R}$ for which the series converges, absolutely converges, or uniformly converges on the half-plane $\mathbb{C}_r := \{z \in \mathbb{C} : \operatorname{Re} z > r\}$, respectively (see, e.g., [5]). It is well known that if we let $L = \limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n}$, then in case $L < \infty$, the following inequalities always hold (see, e.g., [4]):

$$\limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} \leq \sigma_c \leq \sigma_u \leq \sigma_a \leq \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} + L. \quad (2)$$

E-mail addresses: HO0001LU@e.ntu.edu.sg (X. Hou), lhkhoy@ntu.edu.sg (L.H. Khoi).

Set $D = \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n}$. From (2) it follows, for $L < \infty$, that when D is finite, $\sigma_c, \sigma_a, \sigma_u$ are all finite; when $D = \infty$, series (1) does not converge for any $z \in \mathbb{C}$; when $D = -\infty$, $\sigma_c = \sigma_a = \sigma_u = -\infty$, and hence series (1) represents an entire function in \mathbb{C} . Moreover, if a Dirichlet series represents an entire function in \mathbb{C} , we must have $D = -\infty$.

Throughout this Note, the condition $L < \infty$ is always supposed to hold.

1.2. The main goal

Let \mathcal{H} be a space of holomorphic functions on a set $G \subseteq \mathbb{C}$. If an analytic function φ maps G into itself, the *composition operator* C_φ is a linear operator defined by the rule $(C_\varphi f)(z) = f \circ \varphi(z)$, $z \in G$, $f \in \mathcal{H}$.

Composition operators have been investigated on various spaces of holomorphic functions of one variable, as well as in higher dimensions. We refer the reader to the excellent monographs [2,9] for detailed information. In particular, an extensive study of composition operators was carried out on spaces of either entire functions or classical Dirichlet series (see, e.g., [1,3] and references therein).

A natural question to ask is: what about generalized Dirichlet series (1)? The aim of this Note is to study various properties of composition operators on the class of entire Dirichlet series with zero Ritt order and finite logarithmic orders. It seems this topic has never been treated before.

2. Preliminaries and auxiliary results

We consider Dirichlet series (1) which are entire functions in the complex plane \mathbb{C} , that is

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \quad \text{with } D(f) = \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} = -\infty. \quad (3)$$

Note that condition $D = -\infty$ is equivalent to $\lim_{n \rightarrow \infty} |a_n|^{1/\lambda_n} = 0$.

2.1. Different notions of orders for Dirichlet series

Let $f(z)$ be an entire function in \mathbb{C} . The *ordinary order* ρ of f is defined as the greatest lower bound of values of μ such that $M_f(r) < e^{r^\mu}$ for all sufficiently large r , where $M_f(r) = \max_{|z|=r} |f(z)|$, $r > 0$, is its *maximum modulus function*. This order can be computed by $\rho = \limsup_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log r}$.

For an entire Dirichlet series $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$, there is the so-called *Ritt order* (see, e.g., [8]) which is defined as follows: an entire function $f(z)$ is of Ritt order ρ_R if, for every $\varepsilon > 0$, for $-\sigma$ sufficiently large positive, $M(\sigma) < e^{e^{-\sigma(\rho_R + \varepsilon)}}$ and if, besides, there exists a sequence $(\sigma_k)_{k=1}^{\infty} \downarrow -\infty$ for which $M(\sigma_k) > e^{e^{-\sigma_k(\rho_R - \varepsilon)}}$. Here $M(\sigma) = \sup_{t \in \mathbb{R}} |f(\sigma + it)|$, $\sigma \in \mathbb{R}$. The Ritt order can be computed by $\rho_R = \limsup_{\sigma \rightarrow -\infty} \frac{\log \log M(\sigma)}{-\sigma} = \limsup_{n \rightarrow \infty} \frac{\lambda_n \log \lambda_n}{\log \frac{1}{|a_n|}}$.

Furthermore, for entire Dirichlet series with Ritt order 0, Reddy defined the following *logarithmic orders* (see, e.g., [7]): $\rho_*(\mathcal{L}) = \limsup_{\sigma \rightarrow -\infty} \frac{\log \log M(\sigma)}{\log(-\sigma)}$, $\rho_*(\mathcal{L}) = \limsup_{\sigma \rightarrow -\infty} \frac{\log \log \mu(\sigma)}{\log(-\sigma)}$, and $\rho_c(\mathcal{L}) = \limsup_{n \rightarrow \infty} \frac{\log \lambda_n}{\log(\frac{1}{\lambda_n} \log \frac{1}{|a_n|})}$, where $\mu(\sigma) = \max_{n \geq 1} \{|a_n e^{-(\sigma + it)\lambda_n}|\}$ is the *maximum term*. For an entire Dirichlet series with Ritt order 0 we always have $\rho_R(\mathcal{L}) = \rho_*(\mathcal{L}) = \rho_c(\mathcal{L}) + 1 \geq 1$.

The following result probably explains why only entire Dirichlet series with Ritt order zero are in consideration:

Proposition 2.1. For an entire Dirichlet series (3), if $\rho_R \neq 0$, then $\rho_R(\mathcal{L}) = \infty$.

By this proposition, in case $\rho_R \neq 0$, all logarithmic orders are infinity, and hence there is not much information we can get from them.

Remark 2.2. (1) By convention, the *zero function* has order 0. For this zero function, we can easily check that its Ritt order $\rho_R = 0$ and logarithmic order $\rho_*(\mathcal{L}) = 1$.

(2) For an entire Dirichlet series with only *finitely many nonzero coefficients*, those orders are as follows: $\rho(f) \leq 1$; $\rho_R = 0$; $\rho_*(\mathcal{L})(f) \leq 1$.

2.2. Various spaces of Dirichlet series

Let $\mathcal{H}_{\Lambda(L)}^2$ denote the class of Dirichlet series (1) with $\sum_{n=1}^{\infty} |a_n|^2 < +\infty$ and $L < +\infty$. This is a Hilbert space with the inner product

$$\langle f, g \rangle := \sum_{n=1}^{\infty} a_n \bar{b}_n, \quad f, g \in \mathcal{H}_{\Lambda(L)}^2, \quad (4)$$

where $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ and $g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}$.

In this case, as a simple consequence of the Cauchy–Schwarz inequality, we can see that series (1) converges absolutely in the half-plane $\mathbb{C}_{L/2} = \{z \in \mathbb{C} : \operatorname{Re} z > \frac{L}{2}\}$, and hence, represents a holomorphic function in $\mathbb{C}_{L/2}$.

Let us denote by $\mathcal{H}_{\Lambda(L)}^2(E)$ the subset of $\mathcal{H}_{\Lambda(L)}^2$ consisting of entire Dirichlet series (3), i.e. $\mathcal{H}_{\Lambda(L)}^2(E) := \{f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} : D(f) = -\infty\}$. We can easily verify that this set is a normed space with the norm defined by the inner product (4). As is well known, $\mathcal{H}_{\Lambda(L)}^2(E)$ is dense in $\mathcal{H}_{\Lambda(L)}^2$, and hence it is not a complete inner product space.

Furthermore, we introduce the following classes: $\mathcal{H}_{\Lambda(L)}^2(E, 0)$ of entire Dirichlet series whose Ritt order is zero, $\mathcal{H}_{\Lambda(L)}^2(E, 0) := \{f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} : D(f) = -\infty, \rho_R(f) = 0\}$, and $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$ that consists of entire Dirichlet series with Ritt order zero and $\rho_*(\mathcal{L}) < \infty$, $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho) := \{f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} : D(f) = -\infty, \rho_R(f) = 0, \rho_*(\mathcal{L})(f) < \infty\}$.

We can see that $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$ is a subspace of $\mathcal{H}_{\Lambda(L)}^2(E, 0)$, and in its turn, $\mathcal{H}_{\Lambda(L)}^2(E, 0)$ is a subspace of $\mathcal{H}_{\Lambda(L)}^2(E)$. In order to show this, the most important issue is to ensure that the vector addition operation in these classes are well defined. The following results allow us to obtain such a result. They also have their independent interest.

Theorem 2.3. For any two Dirichlet series f and g from $\mathcal{H}_{\Lambda(L)}^2(E)$

- (a) The Ritt order $\rho_R(f + g) \leq \max\{\rho_R(f), \rho_R(g)\}$.
- (b) If $\rho_R(f) \neq \rho_R(g)$, then $\rho_R(f + g) = \max\{\rho_R(f), \rho_R(g)\}$,

and similarly, for any two Dirichlet series f and g from $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$

- (c) The logarithmic order $\rho_*(\mathcal{L})(f + g) \leq \max\{\rho_*(\mathcal{L})(f), \rho_*(\mathcal{L})(g)\}$.
- (d) If $\rho_*(\mathcal{L})(f) \neq \rho_*(\mathcal{L})(g)$, then $\rho_*(\mathcal{L})(f + g) = \max\{\rho_*(\mathcal{L})(f), \rho_*(\mathcal{L})(g)\}$.

Thus both $\mathcal{H}_{\Lambda(L)}^2(E, 0)$ and $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$ are inner product spaces. What about their completeness? It is surprising that the same phenomena as for $\mathcal{H}_{\Lambda(L)}^2(E)$ occurs. We have the following result:

Proposition 2.4. The “smallest” space $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$ is dense in $\mathcal{H}_{\Lambda(L)}^2(E)$.

Thus we have a “chain” of inner product spaces of Dirichlet series

$$\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho) \subset \mathcal{H}_{\Lambda(L)}^2(E, 0) \subset \mathcal{H}_{\Lambda(L)}^2(E) \subset \mathcal{H}_{\Lambda(L)}^2, \tag{5}$$

where the biggest space is a Hilbert space, while the other three smaller spaces are only pre-Hilbert (that is noncomplete) spaces.

Remark 2.5. In Remark 2.2 we already showed that an entire Dirichlet series with finitely many nonzero coefficients has ordinary order at most 1. In case the series consists of a single term, we can get an exact information about its ordinary order. More precisely, given any sequence (λ_n) with $0 \leq \lambda_n \uparrow \infty$ and $L < \infty$, for each $k \in \mathbb{N}$ fixed, the entire Dirichlet series $f_k(z) := e^{-\lambda_k z}$ obviously belongs to the space $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$, and moreover, $\rho(f_k) = 1$.

2.3. Reproducing kernel property

It can be easily verified that the space $\mathcal{H}_{\Lambda(L)}^2$ is a reproducing kernel Hilbert space (RKHS) on $\mathbb{C}_{L/2}$. For this space the reproducing kernel is computed quite simply.

Theorem 2.6. The reproducing kernel for $\mathcal{H}_{\Lambda(L)}^2$ is given by $k_z(w) = \sum_{n=1}^{\infty} e^{-\lambda_n(\bar{z}+w)}$, $z, w \in \mathbb{C}_{L/2}$.

Note also that for the other spaces in the chain (5), no evaluation functional is bounded.

3. Composition operators on $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$

3.1. Ordinary order of Dirichlet series in $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$

Our first aim is to show that every series from $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$ has finite ordinary order. In order to get this we focus on entire Dirichlet series with nonnegative coefficients as well as Ritt order zero and find the relation between their ordinary order ρ and logarithmic orders. Note that the results in Theorem 3.1 below hold for all entire Dirichlet series with nonnegative coefficients and Ritt order zero, not necessarily in $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$.

Theorem 3.1. For those series of $\mathcal{H}_{\Lambda(L)}^2(E, 0)$, whose coefficients are nonnegative, we have $\rho = \rho_R(\Omega)$, and therefore, $\rho = \rho_*(\Omega) = \rho_R(\Omega) = \rho_c(\Omega) + 1$.

From Theorem 3.1 we can arrive at the following result:

Theorem 3.2. Every Dirichlet series in $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$ has finite ordinary order.

Theorem 3.2 allows us to apply Pólya's result [6] on composition of entire functions, which plays an important role in showing the polynomial form of the symbol φ in the main part of the Note.

3.2. Action of the composition operator C_φ

We formulate the first main result of this Note, whose proof of necessity part is inspired by [1,3].

Theorem 3.3. Let $\varphi(z)$ be an entire function on \mathbb{C} . The composition operator $C_\varphi(f) = f \circ \varphi$ maps $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$ into itself if and only if $\varphi(z) = az + b$, $b \in \mathbb{C}$, where $a \geq 1$ satisfies the condition

$$\forall k \in \mathbb{N} \exists N(k) \text{ such that } a = \frac{\lambda_{N(k)}}{\lambda_k}. \quad (6)$$

3.3. Boundedness of the composition operator C_φ

We formulate the second main result of this Note.

Theorem 3.4. Let $\varphi(z)$ be an entire function on \mathbb{C} . The composition operator $C_\varphi(f) = f \circ \varphi$ is bounded on $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$ if and only if $\varphi(z) = az + b$, $b \in \mathbb{C}$, where $a \geq 1$ satisfies condition (6), i.e., $\forall k \in \mathbb{N} \exists N(k)$ such that $a = \frac{\lambda_{N(k)}}{\lambda_k}$, and b satisfies condition

$$\operatorname{Re} b \geq 0. \quad (7)$$

3.4. Examples of $\mathcal{H}_{\Lambda(L)}^2(E, 0, \rho)$ and corresponding $\varphi(z)$

We illustrate the theorem with various examples of the frequencies (λ_n) , for which possible values of a , beside $a = 1$, are discussed.

Example 3.5. (1) For the classical Dirichlet series, that is, when $\lambda_n = \log n$, we have $L = 1$. In this case, any natural number ℓ can work for a .

(2) For $\lambda_n = n$, any value $\ell \in \mathbb{N}$ can work for a .

(3) Consider a geometric sequence (λ_n) with the ratio $q > 1$, given by $\lambda_n = \lambda_1 q^{n-1}$, $\lambda_1 > 0$, $n \geq 1$. For this case any value q^ℓ ($\ell \in \mathbb{N}$) works for a .

A question can be asked: Does there exist a case when $a = 1$ is the only possible value? It turns out that such a case does exist.

Example 3.6. Let $\lambda_n = n!$. Then 1 is the only value a can take.

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