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Statistics

# A test for parameter change in general causal time series using quasi-likelihood estimator

## *Utilisation de la quasi-vraisemblance pour un test de détection de rupture dans les paramètres des processus causaux*

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A B S T R A C T

In this Note, we propose a new procedure to test a change in the parameter of a process $X = (X_t)_{t\in\mathbb{Z}}$ belonging to a class of causal models including AR($\infty$), ARCH($\infty$), TARCH($\infty$), ... models. Two statistics $\widehat{Q}_n^{(1)}$ and $\widehat{Q}_n^{(2)}$ are constructed using the quasi-likelihood estimator (QMLE) of the parameter. Under the null hypothesis that there is no change, each of these statistics converges weakly to a well-known distribution and the maximum diverges to infinity under the alternative of one change. Some simulation results are reported.



R É S U M É

Dans cette Note, nous proposons un nouveau test de détection de rupture dans le paramètre d'un processus $X = (X_t)_{t\in\mathbb{Z}}$ appartenant à une classe de processus causaux contenant les modèles AR($\infty$), ARCH($\infty$), TARCH($\infty$), .... Deux statistiques $\widehat{Q}_n^{(1)}$ et $\widehat{Q}_n^{(2)}$ sont construites en utilisant l'estimateur du maximum de quasi-vraisemblance du paramètre. Sous l'hypothèse nulle selon laquelle aucun changement n'intervient dans le paramètre, chacune de ces statistiques converge vers une distribution connue et le maximum diverge vers l'infini sous l'hypothèse alternative d'une rupture. Quelques résultats de simulations sont présentés.



### Version française abrégée

Détecter les ruptures dans le paramètre d'un processus est un problème crucial en statistique. Plusieurs auteurs ont travaillé sur ce sujet (voir, par exemple, [15,5,4]). Des procédures ont été développées parmi elles, la procédure CUSUM (somme cumulée). Cette procédure a notamment été utilisée pour des tests de détection de rupture dans les covariances et des tests de détection de rupture dans le paramètre d'un modèle GARCH (voir, par exemple, [1,9,12]). Citons aussi la procédure utilisant des approximations de la vraisemblance des observations (voir par exemple [4]). Ces procédures ont été développées dans un cadre paramétrique et les puissances asymptotiques de ces tests sont inconnues.

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Dans cette Note, un test de détection de rupture dans un cadre semi-paramétrique est développé. Nous considérons une classe de processus causaux $\mathcal{M}_T(M,f)$, contenant les modèles classiques tels que AR($\infty$), ARCH($\infty$), TARCH($\infty$), ARMA-GARCH. On suppose qu'un processus $X \in \mathcal{M}_T(M_\theta, f_\theta)$ dépend d'un paramètre $\theta \in \Theta \subset \mathbb{R}^d$ susceptible de changer avec le temps. Ayant observé un échantillon $(X_1,\dots,X_n)$ de $X$, on se propose d'effectuer le test suivant :

$H_0$ : « le paramètre $\theta$ est constant pour $T = \{1,\dots,n\}$ » contre $H_1$ : le paramètre $\theta$ n'est pas constant pour $T = \{1,\dots,n\}$.

Pour $k, k' \in \{1,\dots,n-1\}$ (avec $k \leqslant k'$) soit $\widehat{\theta}_n(X_k,\dots,X_{k'})$ l'estimateur du maximum de quasi-vraisemblance du paramètre $\theta$ calculé sur $\{k,\dots,k'\}$. L'idée de base de notre procédure est la suivante : si le paramètre $\theta \in \Theta$ reste constant au cours du temps, alors $\widehat{\theta}_n(X_1,\dots,X_k)$, $\widehat{\theta}_n(X_{k+1},\dots,X_n)$ et $\widehat{\theta}_n(X_1,\dots,X_n)$ sont tous des estimateurs de $\theta$ et seront asymptotiquement « proches ». Ainsi, les distances $\|\widehat{\theta}_n(X_1,\dots,X_k) - \widehat{\theta}_n(X_1,\dots,X_n)\|$ et $\|\widehat{\theta}_n(X_{k+1},\dots,X_n) - \widehat{\theta}_n(X_1,\dots,X_n)\|$ seront asymptotiquement « faibles ». Nous construisons ainsi une statistique $\widehat{Q}_n$ qui est asymptotiquement finie sous $H_0$ (voir Theorem 1.1) et qui diverge vers l'infini sous l'hypothèse alternative $H_1$ (voir Theorem 1.2). Ce qui montre que notre procédure est consistante en puissance.

Dans la Section 2, nous présentons les résultats de quelques études empiriques. On s'intéresse aux ruptures dans les paramètres des processus AR(1) et GARCH(1, 1). Des comparaisons ont été faites avec les résultats obtenus par Kouamo et al. [11] et ceux obtenus avec la statistique proposée par Kulperger et Yu [12]. Ces comparaisons montrent que notre procédure est meilleure en ce sens qu'elle est largement plus puissante et les niveaux empiriques sont plus proches du seuil fixé pour le modèle AR. Pour le cas GARCH ce niveau empirique décroît et est plus proche du seuil $\alpha = 0.05$ à partir de $n = 1500$. Voir [7] pour d'autres simulations, pour les preuves des théorèmes ainsi qu'une étude plus détaillée de test de détection de rupture dans les modèles causaux.

## 1. Assumptions and main results

We consider a class $\mathcal{M}_T(M,f)$ of causal time series. Let $M, f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be measurable functions, $(\xi_t)_{t\in\mathbf{Z}}$ be a sequence of centered independent and identically distributed (iid) random variables and satisfying $\mathrm{var}(\xi_0) = \sigma^2$ and $\Theta$ a compact subset of $\mathbb{R}^d$. Let $T \subset \mathbf{Z}$, and for any $\theta \in \Theta$, define

**Class $\mathcal{M}_T(M_\theta, f_\theta)$.** The process $X = (X_t)_{t\in\mathbf{Z}}$ belongs to $\mathcal{M}_T(M_\theta, f_\theta)$ if it satisfies the relation:

$$X_{t+1} = M_\theta\big((X_{t-i})_{i\in\mathbb{N}}\big)\xi_t + f_\theta\big((X_{t-i})_{i\in\mathbb{N}}\big) \quad \text{for all } t \in T. \tag{1}$$

See [2] for more details about this class of model. Numerous classical time series (for instance AR($\infty$), ARCH($\infty$), TARCH($\infty$)) are included in $\mathcal{M}_{\mathbf{Z}}(M,f)$.

Now, assume that a trajectory $(X_1,\dots,X_n)$ of $X = (X_t)_{t\in\mathbf{Z}}$ is observed and consider the hypothesis:

**H**$_0$: there exists $\theta_0 \in \Theta$ such that the process $(X_t)_{t\in\mathbf{Z}}$ belongs to the class $\mathcal{M}_{\{1,\dots,n\}}(M_{\theta_0}, f_{\theta_0})$;
**H**$_1$: there exist $\theta_1^*, \theta_2^* \in \Theta$ with $\theta_1^* \neq \theta_2^*$, such that $(X_t)_{t\in\mathbf{Z}}$ belongs to $\mathcal{M}_{T_1^*}(M_{\theta_1^*}, f_{\theta_1^*}) \cap \mathcal{M}_{T_2^*}(M_{\theta_2^*}, f_{\theta_2^*})$ where $T_j^* = \{t_{j-1}^*+1, t_{j-1}^*+2,\dots,t_j^*\}$ for $j = 1,2$ with $t_0^* = 0$, $t_2^* = n$ and $t_1^* = [n\tau^*]$ for some $\tau^* \in (0,1)$.

It is easy to see that under $\mathrm{H}_1$ the stationarity property is lost after the change. This is not the case in many existing works (for instance Kouamo et al. [11]) where the stationarity or the $K$-th order stationarity after the change is an essential assumption. See for instance [6,10,4,13,12,14,1,3] for more references about change-point problems.

### 1.1. Assumptions on the class of models $\mathcal{M}_{\mathbf{Z}}(f_\theta, M_\theta)$

We will use the following norms:

(i) $\|\cdot\|$ applied to a vector denotes the Euclidean norm of the vector;
(ii) for any compact set $U \subseteq \mathbb{R}^d$ and for any $g : U \to \mathbb{R}^{d'}$, $\|g\|_U = \sup_{\theta\in U}(\|g(\theta)\|)$.

Throughout the sequel, we will assume that the functions $\theta \mapsto M_\theta$ and $\theta \mapsto f_\theta$ are twice continuously differentiable on $\Theta$. Let $\Psi_\theta = f_\theta, M_\theta$ and $i = 0,1,2$, then define

**Assumption A$_i(\Psi_\theta, \Theta)$.** Assume that $\|\partial^i \Psi_\theta(0)/\partial\theta^i\|_\Theta < \infty$ and there exists a sequence of non-negative real numbers $(\alpha_k^{(i)}(\Psi_\theta,\Theta))_{k\geqslant 1}$ such that $\sum_{k=1}^{\infty} \alpha_k^{(i)}(\Psi_\theta,\Theta) < \infty$ satisfying

$$\left\| \frac{\partial^i \Psi_\theta(x)}{\partial\theta^i} - \frac{\partial^i \Psi_\theta(y)}{\partial\theta^i} \right\|_\Theta \leqslant \sum_{k=1}^{\infty} \alpha_k^{(i)}(\Psi_\theta,\Theta)|x_k - y_k| \quad \text{for all } x = (x_k)_{k\in\mathbb{N}} \text{ and } y = (y_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}.$$

The sequel refers to the particular case called "ARCH-type process" if $f_\theta = 0$ and if the following assumption holds with $h_\theta = M_\theta^2$:

**Assumption $\mathbf{A}_i(h_\theta, \Theta)$.** Assume that $\|\partial^i h_\theta(0)/\partial\theta^i\|_\Theta < \infty$ and there exists a sequence of non-negative real numbers $(\alpha_k^{(i)}(h_\theta,\Theta))_{k\geqslant 1}$ such as $\sum_{k=1}^{\infty}\alpha_k^{(i)}(h_\theta,\Theta) < \infty$ satisfying

$$\left\|\frac{\partial^i h_\theta(x)}{\partial\theta^i} - \frac{\partial^i h_\theta(y)}{\partial\theta^i}\right\|_\Theta \leqslant \sum_{k=1}^{\infty}\alpha_k^{(i)}(h_\theta,\Theta)\left|x_k^2 - y_k^2\right| \quad \text{for all } x = (x_k)_{k\in\mathbb{N}} \text{ and } y = (y_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}.$$

Then define the set:

$$\Theta(r) := \left\{\theta \in \Theta,\ A_0\big(f_\theta,\{\theta\}\big) \text{ and } A_0\big(M_\theta,\{\theta\}\big) \text{ hold with } \sum_{k\geqslant 1}\alpha_k^{(0)}(f_\theta,\theta) + \big(E|\xi_0|^r\big)^{1/r}\sum_{k\geqslant 1}\alpha_k^{(0)}(M_\theta,\theta) < 1\right\}$$
$$\cup\left\{\theta\in\Theta,\ f_\theta = 0 \text{ and } A_0\big(h_\theta,\{\theta\}\big) \text{ holds with } \big(E|\xi_0|^r\big)^{2/r}\sum_{k\geqslant 1}\alpha_k^{(0)}(h_\theta,\theta) < 1\right\}.$$

The Lipschitz-type hypothesis $A_i(\Psi_\theta,\Theta)$ is classical for the existence of solutions of the model. If $\theta\in\Theta(r)$ the existence of a unique causal, stationary and ergodic solution $X = (X_t)_{t\in\mathbf{Z}} \in \mathcal{M}_{\mathbf{Z}}(f_\theta, M_\theta)$ is ensured (see [2]).

The following assumptions are needed to study QMLE property.

**Assumption $\mathbf{D}(\Theta)$.** $\exists \underline{h} > 0$ such that $\inf_{\theta\in\Theta}(|h_\theta(x)|) \geqslant \underline{h}$ for all $x\in\mathbb{R}^{\mathbb{N}}$.

**Assumption $\mathbf{Id}(\Theta)$.** For all $\theta,\theta'\in\Theta$,

$$\big(f_\theta(X_0,X_{-1},\ldots) = f_{\theta'}(X_0,X_{-1},\ldots) \text{ and } h_\theta(X_0,X_{-1},\ldots) = h_{\theta'}(X_0,X_{-1},\ldots) \text{ a.s.}\big) \quad\Longrightarrow\quad \theta = \theta'.$$

**Assumption $\mathbf{Var}(\Theta)$.** For all $\theta\in\Theta$, one of the families $(\frac{\partial f_\theta}{\partial\theta^i}(X_0,X_{-1},\ldots))_{1\leqslant i\leqslant d}$ or $(\frac{\partial h_\theta}{\partial\theta^i}(X_0,X_{-1},\ldots))_{1\leqslant i\leqslant d}$ is a.s. linearly independent.

**Assumption $\mathbf{K}(f_\theta, M_\theta, \Theta)$.** For $i = 0,1,2$, $\mathbf{A}_i(f_\theta,\Theta)$ and $\mathbf{A}_i(M_\theta,\Theta)$ (or $\mathbf{A}_i(h_\theta,\Theta)$) hold and there exists $\ell > 2$ such that $\alpha_j^{(i)}(f_\theta,\Theta) + \alpha_j^{(i)}(M_\theta,\Theta) + \alpha_j^{(i)}(h_\theta,\Theta) = \mathcal{O}(j^{-\ell})$, for $i = 0,1$, with the convention that if $\mathbf{A}_i(M_\theta,\Theta)$ holds then $\alpha_j^{(i)}(h_\theta,\Theta) = 0$ and if $\mathbf{A}_i(h_\theta,\Theta)$ holds then $\alpha_j^{(i)}(M_\theta,\Theta) = 0$.

Assume that a trajectory $(X_1,\ldots,X_n)$ is observed. If $(X_1,\ldots,X_n)\in\mathcal{M}_{\{1,\ldots,n\}}(M_\theta,f_\theta)$, then for $T\subset\{1,\ldots,n\}$, the conditional quasi-(log)likelihood computed on $T$ is given by:

$$L_n(T,\theta) := -\frac{1}{2}\sum_{t\in T} q_t(\theta) \quad \text{with } q_t(\theta) = \frac{(X_t - f_\theta^t)^2}{h_\theta^t} + \log\big(h_\theta^t\big)$$

where $f_\theta^t = f_\theta(X_{t-1},X_{t-2}\ldots)$, $M_\theta^t = M_\theta(X_{t-1},X_{t-2}\ldots)$ and $h_\theta^t = {M_\theta^t}^2$. As is usually done now (see [2]), we approximate this conditional log-likelihood by:

$$\widehat{L}_n(T,\theta) := -\frac{1}{2}\sum_{t\in T}\widehat{q}_t(\theta) \quad \text{with } \widehat{q}_t(\theta) := \frac{(X_t - \widehat{f}_\theta^t)^2}{\widehat{h}_\theta^t} + \log\big(\widehat{h}_\theta^t\big)$$

where $\widehat{f}_\theta^t = f_\theta(X_{t-1},\ldots,X_1,0,0,\ldots)$, $\widehat{M}_\theta^t = M_\theta(X_{t-1},\ldots,X_1,0,0,\ldots)$ and $\widehat{h}_\theta^t = (\widehat{M}_\theta^t)^2$. For $T\subset\{1,\ldots,n\}$, define the quasi-likelihood estimator computed on $T$ by $\widehat{\theta}_n(T) := \operatorname{argmax}_{\theta\in\Theta}(\widehat{L}_n(T,\theta))$. Now, for $T\subset\{1,\ldots,n\}$ define

$$\widehat{G}_n(T) := \frac{1}{Card(T)}\sum_{t\in T}\left(\frac{\partial\widehat{q}_t(\widehat{\theta}_n(T))}{\partial\theta}\right)\left(\frac{\partial\widehat{q}_t(\widehat{\theta}_n(T))}{\partial\theta}\right)' \quad\text{and}\quad \widehat{F}_n(T) := \frac{1}{Card(T)}\sum_{t\in T}\frac{\partial^2\widehat{q}_t(\widehat{\theta}_n(T))}{\partial\theta\,\partial\theta'}.$$

For $k = 1,\ldots,n-1$, denote $T_k = \{1,\ldots,k\}$, $\overline{T}_k = \{k+1,\ldots,n\}$ and define

$$\widehat{\Sigma}_{n,k} := \frac{k}{n}\widehat{F}_n(T_k)\widehat{G}_n(T_k)^{-1}\widehat{F}_n(T_k)\mathbf{1}_{\det(\widehat{G}_n(T_k))\neq 0} + \frac{n-k}{n}\widehat{F}_n(\overline{T}_k)\widehat{G}_n(\overline{T}_k)^{-1}\widehat{F}_n(\overline{T}_k)\mathbf{1}_{\det(\widehat{G}_n(\overline{T}_k))\neq 0}.$$

For $k = 1,\ldots,n-1$, $\widehat{\Sigma}_{n,k}$ is symmetric positive semi-definite. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence satisfying $v_n\to\infty$ and $v_n/n\to 0$ (as $n\to\infty$). Denote $\Pi_n = [v_n, n - v_n]\cap\mathbb{N}$ and define the statistics:

$$\widehat{Q}_n^{(1)} := \max_{k\in\Pi_n} \widehat{Q}_{n,k}^{(1)} \quad \text{where } \widehat{Q}_{n,k}^{(1)} := \frac{k^2}{n}\big(\widehat{\theta}_n(T_k) - \widehat{\theta}_n(T_n)\big)' \widehat{\Sigma}_{n,k}\big(\widehat{\theta}_n(T_k) - \widehat{\theta}_n(T_n)\big);$$

$$\widehat{Q}_n^{(2)} := \max_{k\in\Pi_n} \widehat{Q}_{n,k}^{(2)} \quad \text{where } \widehat{Q}_{n,k}^{(2)} := \frac{(n-k)^2}{n}\big(\widehat{\theta}_n(\overline{T}_k) - \widehat{\theta}_n(T_n)\big)' \widehat{\Sigma}_{n,k}\big(\widehat{\theta}_n(\overline{T}_k) - \widehat{\theta}_n(T_n)\big).$$

The new test statistic is defined by

$$\widehat{Q}_n := \max\big(\widehat{Q}_n^{(1)}, \widehat{Q}_n^{(2)}\big).$$

We evaluated the procedure with $v_n = [\log n], [(\log n)^2], [(\log n)^3]$ from numerical simulations and recommend to use $v_n = [(\log n)^2]$ for linear model and $v_n = [(\log n)^{5/2}]$ for GARCH-type model.

### *1.2. Main results*

**Theorem 1.1.** *Assume* $\mathbf{D}(\Theta)$, $\mathbf{Id}(\Theta)$, **Var** *and* $\mathbf{K}(f_\theta, M_\theta, \Theta)$. *Under the null hypothesis* $\mathrm{H}_0$, *if* $\theta_0 \in \mathring{\Theta}(4)$ *and denoting* $W_d$ *a d-dimensional Brownian bridge, then for* $j = 1, 2$,

$$\widehat{Q}_n^{(j)} \xrightarrow[n\to\infty]{\mathcal{D}} \sup_{0\leqslant\tau\leqslant1} \big\|W_d(\tau)\big\|^2.$$

The proof of this result and the one of Theorem 1.2 is provided in [7]. For any $\alpha \in (0,1)$, let $C_\alpha$ denote the $(1-\alpha/2)$-quantile of the distribution of $\sup_{0\leqslant\tau\leqslant1} \|W_d(\tau)\|^2$. Then,

**Corollary 1.** *Under assumptions of Theorem* 1.1:

$$\textit{for any } \alpha \in (0,1), \quad \limsup_{n\to\infty} P(\widehat{Q}_n > C_\alpha) \leqslant \alpha.$$

The quantile values of the distribution of $\sup_{0\leqslant\tau\leqslant1} \|W_d(\tau)\|^2$ are known (see for instance Kiefer [8] for $d \in \{1,\dots,5\}$). Theorem 1.1 and Corollary 1 imply that a large value of $\widehat{Q}_n$ means there is a change in the model. At a nominal level $\alpha$, the critical region of the test is $(\widehat{Q}_n > C_\alpha)$. Fig. 1 is an illustration of the test procedure for AR(1) process. Figs. 1(a) and 1(b) show that the values of $\widehat{Q}_{n,k}^{(1)}$ and $\widehat{Q}_{n,k}^{(2)}$ are all below the horizontal line which represents the limit of the critical region at the level $\alpha = 0.05$. Figs. 1(c) and 1(d) show that $\widehat{Q}_{n,k}^{(1)}$ and $\widehat{Q}_{n,k}^{(2)}$ are larger and increase around the point where the change occurs.

The asymptotic behavior under the alternative $\mathrm{H}_1$ is given by Theorem 1.2.

**Theorem 1.2.** *Assume* $\mathbf{D}(\Theta)$, $\mathbf{Id}(\Theta)$, **Var** *and* $\mathbf{K}(f_\theta, M_\theta, \Theta)$. *Under* $\mathrm{H}_1$, *if* $\theta_1^*, \theta_2^* \in \mathring{\Theta}(4)$, *then*

$$\widehat{Q}_n \xrightarrow[n\to\infty]{a.s.} \infty.$$

**Remark 1.**

(1) Theorem 1.2 shows that the test is consistent in power.
(2) This procedure can also be used to test multiple changes using ICSS type algorithm developed by Inclán and Tiao [6].

## 2. Some empirical results

We evaluate in this section the numerical performances of the test procedure and compare our results with those obtained by Kouamo et al. [11] and the results obtained from the residual CUSUM test by using the statistics defined by Kulperger and Yu [12]. In the following models, $(\xi_t)_{t\in\mathbf{Z}}$ are iid standard Gaussian random variables.

### *2.1. Test for parameter change in AR(1) models*

Consider an AR(1) process: $X_t = \phi^* X_{t-1} + \xi_t$ with $\phi^* \in \Theta$ where $\Theta = \{\theta = \phi \in \mathbb{R}/|\phi| < 1\}$. At the level $\alpha = 0.05$, the critical value of the test is $C_\alpha \simeq 2.20$. For $n = 1024, 2048, 4096$; we generate a sample $(X_1,\dots,X_n)$ in the following situations: (i) the parameter $\theta_0 = 0.9$ remains constant and (ii) the parameter $\theta_0 = 0.9$ changes to $\theta_1 = 0.5$ at $n/2$. Table 1 indicates the proportion of the number of rejections of the null hypothesis out of 100 repetitions.

As seen in Table 1, the empirical levels are closer to the nominal level 0.05 than those obtained by Kouamo et al. and that our test procedure is more powerful.

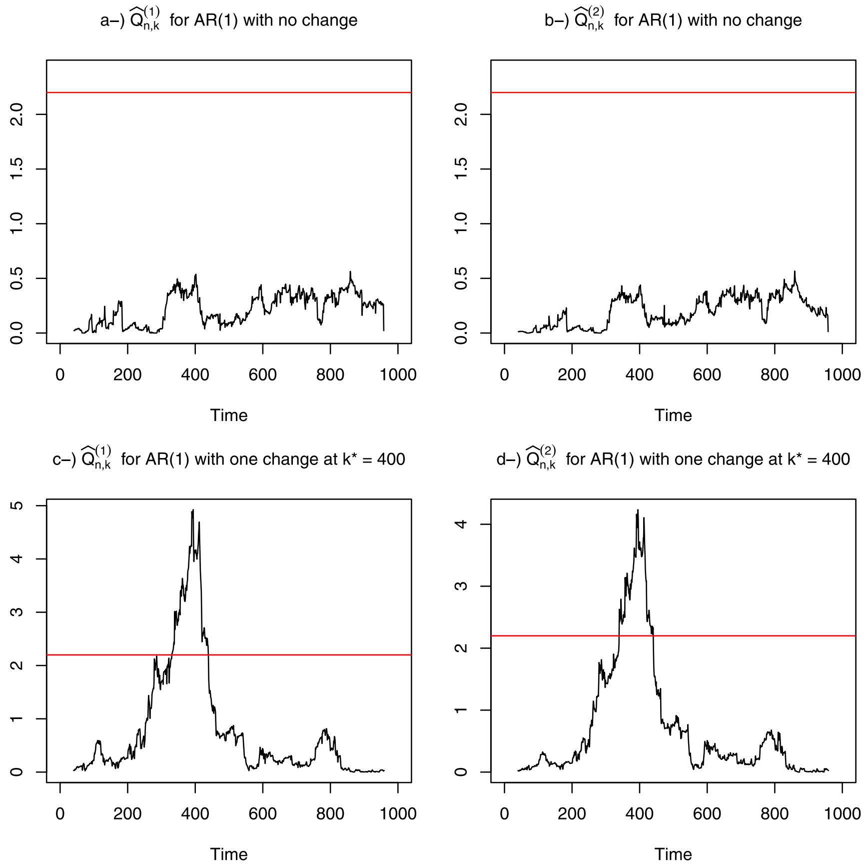


**Fig. 1.** The statistics $\widehat{Q}^{(1)}_{n,k}$ and $\widehat{Q}^{(2)}_{n,k}$ computed for 1000 sample of AR(1) with $v_n = [(\log n)^2]$. (a) and (b): the parameter $\phi_1 = 0.3$ remains constant. (c) and (d): the parameter $\phi_1 = 0.3$ changing to 0.5 at $k^* = 400$.

**Table 1**
Empirical levels and powers at nominal level 0.05 of the test for parameter change in AR(1) model. The empirical levels are computed when $\theta_0 = 0.9$; the empirical powers are computed when $\theta_0$ changes to $\theta_1 = 0.5$ at $n/2$. Figures in brackets are the results obtained by Kouamo et al. [11] at the scale $J = 4$ with KSM and CVM statistic in wavelet domain.

| | $n = 1024$ | $n = 2048$ | $n = 4096$ |
|---|---|---|---|
| Empirical levels | 0.080 (0.134 ; 0.092) | 0.070 (0.100 ; 0.062) | 0.050 (0.082 ; 0.040) |
| Empirical powers when $\theta_1 = 0.5$ | 0.980 (0.590 ; 0.530) | 0.990 (0.720 ; 0.680) | 0.990 (0.810 ; 0.790) |

**Table 2**
Empirical levels and powers at nominal level 0.05 of the test for parameter change in GARCH$(1,1)$ model. The empirical levels are computed when $\theta_0 = (1, 0.4, 0.1)$; the empirical powers are computed when $\theta_0$ changes to $\theta_1$ at $n/2$. Figures in brackets are the results of the residual CUSUM test using $CUSUM^{(2)}$ statistic defined by Kulperger and Yu [12].

| | $n = 500$ | $n = 1000$ | $n = 1500$ |
|---|---|---|---|
| Empirical levels | 0.100 (0.030) | 0.078 (0.032) | 0.052 (0.042) |
| Empirical powers when $\theta_1 = (0.7, 0.4, 0.1)$ | 0.498 (0.334) | 0.752 (0.658) | 0.934 (0.848) |
| Empirical powers when $\theta_1 = (1, 0.4, 0.3)$ | 0.654 (0.404) | 0.968 (0.772) | 0.976 (0.922) |

### *2.2. Test for parameter change in GARCH$(1,1)$ models*

Consider the GARCH$(1,1)$ model defined by: $\forall t \in \mathbf{Z}$, $X_t = \sigma_t \xi_t$ where $\sigma_t^2 = \alpha_0^* + \alpha_1^* X_{t-1}^2 + \beta_1^* \sigma_{t-1}^2$, with $\theta_0^* = (\alpha_0^*, \alpha_1^*, \beta_1^*) \in \Theta \subset ]0,\infty[ \times [0,\infty[^2$ satisfying $\alpha_1^* + \beta_1^* < 1$. The ARCH$(\infty)$ representation is $\sigma_t^2 = \alpha_0^*/(1-\beta_1^*) + \alpha_1^* \sum_{k\geqslant 1} (\beta_1^*)^{k-1} X_{t-k}^2$.

At level $\alpha = 0.05$, the critical value is $C_\alpha \simeq 3.47$. For $n = 500, 1000, 1500$, we generate a sample $(X_1, \ldots, X_n)$ in the following situations: (i) the parameter $\theta_0 = (1, 0.4, 0.1)$ remains constant and (ii) the parameter $\theta_0 = (1, 0.4, 0.1)$ changes to $\theta_1$ (see Table 2) at $n/2$. Table 2 indicates the proportion of the number of rejections of the null hypothesis out of 500 repetitions.

Once again, Table 2 shows that the results obtained with our test statistic $\widehat{Q}_n$ are more accurate.

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