



Partial Differential Equations

Existence of global strong solutions for the barotropic Navier–Stokes system with large initial data on the rotational part of the velocity

Existence de solutions fortes globales pour le système de Navier–Stokes compressible avec des données initiales grandes sur la partie rotationnelle de la vitesse

Boris Haspot

Ceremade, UMR CNRS 7534, université de Paris Dauphine, place du Maréchal DeLattre De Tassigny, 75775 Paris cedex 16, France

ARTICLE INFO

Article history:

Received 20 March 2012

Accepted 26 April 2012

Available online 12 May 2012

Presented by the Editorial Board

ABSTRACT

We show the existence of global strong solutions for the compressible Navier–Stokes system in dimension $N \geq 2$ with large initial data on the rotational part of the velocity. By following Chemin and Gallagher (2009, 2011) [3,4], we aim at exhibiting large initial data u_0 such that the projection on the divergence field $\mathbb{P}u_0$ is large in $B_{\infty,\infty}^{-1}$ (which is the largest space invariant by the scaling of the equations) and such that these initial data generate global strong solution. The fact that the smallness hypothesis in Chemin and Gallagher (2009) [3] holds on the nonlinear term of convection enables us to split the solution of the compressible Navier–Stokes equations in the sum of an incompressible solution and of a purely compressible solution. Combining the notion of *quasi-solution* introduced in Haspot [8,9,7], we obtain the existence of global strong solution for the shallow water system for large initial velocity both on the irrotational and rotational part.

© 2012 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Nous montrons l'existence de solutions fortes globales pour le système de Navier–Stokes compressible en dimension $N \geq 2$ avec des données initiales grandes sur la partie rotationnelle de la vitesse. Suivant Chemin et Gallagher (2009, 2011) [3,4], nous cherchons à exhiber des données initiales u_0 telles que la projection sur les champs de vecteurs à divergence nulle $\mathbb{P}u_0$ soient grandes dans $B_{\infty,\infty}^{-1}$ (qui est le plus large espace invariant par le scaling des équations) et telle que ces données initiales génèrent des solutions fortes globales. Le fait que l'hypothèse de petitesse dans Chemin et Gallagher (2009) [3] a lieu sur le terme non linéaire de convection nous permet de décomposer la solution des équations de Navier–Stokes compressible comme la somme d'une vitesse incompressible et d'une vitesse purement compressible. Combinant la notion de *quasi-solution* introduite dans Haspot [8,9,7], nous obtenons l'existence de solutions fortes globales avec des données initiales à la fois grande pour la partie irrotationnelle et la partie rotationnelle.

© 2012 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

E-mail address: haspot@ceremade.dauphine.fr.

Version française abrégée

Nous esquissons dans cette note la preuve de l'existence globale de solutions fortes pour le système de Navier–Stokes compressible modélisant un fluide compressible, les équations prennent la forme suivante :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho)D(u)) - \nabla(\lambda(\rho) \operatorname{div} u) + \nabla P(\rho) = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases} \quad (1)$$

Ici $u = u(t, x) \in \mathbb{R}^N$ avec $N \geq 2$ correspond à la vitesse du liquide, $\rho = \rho(t, x) \in \mathbb{R}^+$ sa densité et on a $D(u) = \frac{1}{2}(\nabla u + {}^t \nabla u)$. La pression P s'écrit $P(\rho) = a\rho^\gamma$ avec $a > 0$, $\gamma \geq 1$. $\mu(\rho) > 0$ et $\lambda(\rho) + \mu(\rho) > 0$ sont les coefficients de viscosité. Avant d'énoncer les théorèmes on note $\rho_{HFl} = \sum_{k \geq l} \Delta_k \rho$ et $\rho_{BFl} = \sum_{k < l} \Delta_k \rho$ avec $l \geq 0$ (on réfère à [5] pour les définitions liées à la théorie de Littlewood–Paley) et $q = \rho - \bar{\rho}$ avec $\bar{\rho} > 0$.

Théorème 0.1. *On suppose les coefficients de viscosités constants. Soit des données initiales telles que $q_0 \in \tilde{B}_{2,1}^{\frac{N}{2}-1, \frac{N}{2}}$, $\rho_0 \geq c > 0$ et $u_0 = u_0^1 + u_0^2$ telles que $u_0^1 \in B_{2,2}^{\frac{N}{2}-1} \cap B_{\infty,1}^{-1}$ avec $\operatorname{div} u_0^1 = 0$ et $u_0^2 \in B_{2,1}^{\frac{N}{2}-1}$. Il existe deux constantes C et ε telles que si :*

$$\|\mathbb{P}(e^{t\mu\Delta} u_0^1 \cdot \nabla e^{t\mu\Delta} u_0^1)\|_{\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} \leq C^{-1} \exp(-C \|u_0^1\|_{B_{\infty,2}^{-1}}^4) \quad (2)$$

et :

$$\|q_0\|_{\tilde{B}_{2,1}^{\frac{N}{2}-1, \frac{N}{2}}} + \|u_0^2\|_{B_{2,1}^{\frac{N}{2}-1}} \leq \varepsilon,$$

alors il existe une solution forte globale (ρ, u) du système (1) telle que : $u = u^1 + u^2$ et $\rho = q + \bar{\rho}$ avec :

$$q \in \tilde{C}(\mathbb{R}^+, \tilde{B}_{2,1}^{\frac{N}{2}-1, \frac{N}{2}}) \cap L^1(\mathbb{R}^+, \tilde{B}_{2,1}^{\frac{N}{2}+1, \frac{N}{2}}) \quad \text{et} \quad u^2 \in \tilde{C}(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}-1}) \cap L^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}+1}),$$

et :

$$u^1 \in \tilde{C}(\mathbb{R}^+, B_{2,2}^{\frac{N}{2}-1}) \cap L^1(\mathbb{R}^+, B_{2,2}^{\frac{N}{2}+1}) \cap \tilde{L}^\infty(\mathbb{R}^+, B_{\infty,1}^{-1}) \cap \tilde{L}^1(\mathbb{R}^+, B_{\infty,1}^1).$$

On réfère à [5] pour les notations sur les espaces de Besov.

Remarque 1. Suivant [3], on peut obtenir des données initiales oscillantes u_0^1 vérifiant (2) et grandes dans $B_{\infty,\infty}^{-1}$ qui est le plus large espace invariant pour le scaling des équations de Navier–Stokes incompressible.

Esquissons la preuve. Elle se divise en deux parties.

Solution incompressible : On commence par résoudre le système de Navier–Stokes incompressible avec donnée initiale u_0^1 , c'est à dire :

$$\begin{cases} \partial_t u^1 + u^1 \cdot \nabla u^1 - \mu \Delta u^1 + \nabla \Pi^1 = 0, \\ \operatorname{div} u^1 = 0, \quad u^1|_{t=0} = u_0^1. \end{cases} \quad (3)$$

On peut alors montrer l'existence de solution fortes globales au système (3) sous l'hypothèse (2). En effet il s'agit d'appliquer un théorème du point fixe dans $E = \tilde{C}(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}-1}) \cap \tilde{L}^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}+1})$ pour \bar{u} lorsque l'on décompose la solution sous la forme $u^1 = e^{\mu t \Delta} u_0^1 + \bar{u}$. On vérifie aisément que :

$$\bar{u} = \int_0^t e^{\mu(t-s)\Delta} (\mathbb{P}(e^{t\mu\Delta} u_0^1 \cdot \nabla e^{t\mu\Delta} u_0^1) + \mathbb{P}(e^{t\mu\Delta} u_0^1 \cdot \nabla \bar{u}) + \mathbb{P}(\bar{u} \cdot \nabla e^{t\mu\Delta} u_0^1) + \mathbb{P}(\bar{u} \cdot \nabla \bar{u}))(s) ds.$$

Pour pouvoir appliquer le théorème du point fixe, on a besoin d'une condition de petitesse sur le terme source $\int_0^t e^{\mu(t-s)\Delta} \mathbb{P}(e^{t\mu\Delta} u_0^1 \cdot \nabla e^{t\mu\Delta} u_0^1) ds$ ou plus précisément sur $\mathbb{P}(e^{t\mu\Delta} u_0^1 \cdot \nabla e^{t\mu\Delta} u_0^1)$ dans $\tilde{L}^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}+1})$ (en effet on utilise des résultats classiques nous donnant des estimations dans des espaces de Besov pour l'équation de la chaleur et le fait que même si u_0^1 est seulement dans $B_{2,2}^{\frac{N}{2}-1}$ on a cependant un gain de régularité sur le terme de convection avec $\mathbb{P}(e^{t\mu\Delta} u_0^1 \cdot \nabla e^{t\mu\Delta} u_0^1)$ est dans $\tilde{L}^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}+1})$ voir [2]). Cette condition de petitesse se traduit en particulier par la condition (2) après une application du lemme de Gronwall. Pour plus de détails sur l'application du point fixe dans E nous renvoyons à [3].

Solution globale : Il s'agit ensuite de vérifier que l'on peut trouver une solution du système (1) lorsque l'on écrit la solution sous la forme $u = u^1 + u^2$. Pour ce faire on montre l'existence de solutions fortes globales pour le système vérifié par (q, u^2) en supposant q_0 et u_0^2 petits respectivement dans $\tilde{B}_{2,1}^{\frac{N}{2}-1, \frac{N}{2}}$ et dans $B_{2,1}^{\frac{N}{2}-1}$. L'idée consiste à appliquer les mêmes arguments que dans [5] en travaillant autour de la solution u^1 , on rappelle qu'une des difficultés majeures consiste à exhiber un effet d'amortissement sur la densité q . La seconde difficulté consiste à observer que le terme $\nabla \Pi^1$ qui interviendra comme terme source dans l'équation du moment vérifiée par u^2 est petit dans $\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})$ et ceci grâce à la condition (2) qui permet de montrer après application de l'opérateur div à (3) que $\Delta \Pi^1$ est petit dans $\tilde{L}^1(B_{2,1}^{\frac{N}{2}-2})$. On peut ensuite suivre les étapes de [5].

1. Introduction

The motion of a general barotropic compressible fluid is described by the following system:

$$\begin{cases} \partial_t \rho + \text{div}(\rho u) = 0, \\ \partial_t(\rho u) + \text{div}(\rho u \otimes u) - \text{div}(\mu(\rho)D(u)) - \nabla(\lambda(\rho) \text{div} u) + \nabla P(\rho) = \rho f, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{4}$$

Here $u = u(t, x) \in \mathbb{R}^N$ stands for the velocity field, $\rho = \rho(t, x) \in \mathbb{R}^+$ is the density and $D(u) = \frac{1}{2}(\nabla u + {}^t \nabla u)$. The pressure P is such that $P(\rho) = a\rho^\gamma$ with $\gamma \geq 1$. We denote by $\mu(\rho) > 0$ and $\mu(\rho) + \lambda(\rho) > 0$ the viscosity coefficients of the fluid. In the sequel we shall consider the shallow water system when $\mu(\rho) = \mu\rho$ with $\mu > 0$ and $\lambda(\rho) = 0$. In the case of barotropic fluids, it is easy to see that the transformations: $(\rho(t, x), u(t, x)) \rightarrow (\rho(l^2 t, lx), lu(l^2 t, lx))$, $l \in \mathbb{R}$, have a property of invariance by scaling, provided that the pressure term has been changed in $\lambda^2 P$. The use of critical functional frameworks led to several new well-posedness results for compressible fluids of global strong solution with small initial data (see [1,2,5, 8,9]). However the existence of global strong solution with large initial data remains open. Here we would give now a first kind of answer to this problem for a family of initial velocity with a large irrotational and rotational part. In particular we are going to work around an irrotational *quasi-solution* of the system (1) (introduced in [8,9]) and a purely incompressible solution. We recall that a *quasi-solution* of the shallow water system correspond to a solution of the form $(\rho^1, -\mu \nabla \ln \rho^1)$. Indeed when $P = 0$ it is an exact solution of the shallow water system, furthermore ρ^1 verifies a heat equation $\partial_t \rho^1 - \mu \Delta \rho^1 = 0$. It seems then natural to work around this particular solution in order to obtain global strong solution with large initial data for (4), however we observe that this *quasi-solution* is purely irrotational. In order to get the existence of global strong solution with large initial velocity both on the irrotational part and the rotational part, we search to write the initial velocity under the form: $u_0 = -\mu \nabla \ln \rho_0^1 + u_0^2 + u_0^3$ with $\text{div} u_0^2 = 0$ where u_0^2 is purely incompressible and u_0^3 is a remainder velocity and with $\rho_0 = \rho_0^1 + h_0$. More precisely we now search solution of the form $\rho = \rho^1 + h = q^1 + \bar{\rho} + h$ and $u = -\mu \nabla \ln \rho^1 + u^2 + u^3$ and we have when for simplifying $P(\rho) = a\rho^2$ ($u^1 = -\mu \nabla \ln \rho^1$, $v = u^2 + u^3$ and $w = u^1 + u^2$):

$$\begin{cases} \partial_t u^2 + u^1 \cdot \nabla u^2 + u^2 \cdot \nabla u^2 - \mu \Delta u^2 + \nabla \Pi^2 = -u^2 \cdot \nabla u^1, \\ \text{div} u^2 = 0, \\ (u^2)_{/t=0} = u_0^2 \end{cases} \tag{5}$$

and

$$\begin{cases} \partial_t h + u \cdot \nabla h + \text{div} u^3 = -v \cdot \nabla q^1 - h \text{div} u^3 - q^1 \text{div} u^3, \\ \partial_t u^3 + u \cdot \nabla u^3 - \mu \Delta u^3 + a \nabla h = -a \nabla q^1 + \nabla \Pi^2 - u^3 \cdot \nabla w + \mu \nabla \ln \rho \cdot Dv + \mu \nabla (\ln \rho - \ln \rho^1) \cdot Du^1, \\ (h, u^3)_{/t=0} = (h_0, u_0^3). \end{cases} \tag{6}$$

Our goal now consists in solving the systems (5) and (6) with large initial data on u_0^1 in $B_{2,1}^{\frac{N}{2}-1}$ as in [8,9] and on u_0^2 at least in $B_{\infty, \infty}^{-1}$ as in [3] and with small initial data on u_0^3 . To solve (5) we are going to use some ideas developed in [3,4] and in order to obtain global strong solution for the system (6) we are following [5,8,9]. We can now state our main result.

Theorem 1.1. *Let $\mu(\rho) = \mu\rho$ with $\mu > 0$ and $\lambda(\rho) = 0$. Let $\rho_0 = \rho_0^1 + h_0 = \bar{\rho} + q_0^1 + h_0$, $\rho_0 \geq c > 0$ and $u_0 = -\mu \nabla \ln \rho_0^1 + u_0^2 + u_0^3$ with $\text{div} u_0^2 = 0$. Furthermore we assume that $(q_0^1)_{BFI} \in B_{2,1}^{\frac{N}{2}-2}$, $(q_0^1)_{HFI} \in B_{2,1}^{\frac{N}{2}}$, $u_0^2 \in B_{2,1}^{\frac{N}{2}-1}$, $h \in \tilde{B}_{2,1}^{\frac{N}{2}-1, \frac{N}{2}}$ and $u_0^3 \in B_{2,1}^{\frac{N}{2}-1}$. Then it exists $C > 0$ and $l > 0$ large enough depending both on $\|q_0^1\|_{B_{2,1}^{\frac{N}{2}}}$ and ε such that if:*

$$\|(q_0^1)_{BFI}\|_{B_{2,1}^{\frac{N}{2}-2}} \leq \varepsilon \quad \text{and} \quad \|h\|_{\tilde{B}_{2,1}^{\frac{N}{2}-1, \frac{N}{2}}} + \|u_0^3\|_{B_{2,1}^{\frac{N}{2}-1}} \leq \varepsilon, \tag{7}$$

and

$$\begin{aligned} & \left\| \mathbb{P}(\nabla \ln(e^{t\mu\Delta} \rho_0^1) \cdot \nabla e^{t\mu\Delta} u_0^2 + e^{t\mu\Delta} u_0^2 \cdot \nabla \nabla \ln(e^{t\mu\Delta} \rho_0^1) + e^{t\mu\Delta} u_0^2 \cdot \nabla e^{t\mu\Delta} u_0^2) \right\|_{\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} \\ & \leq C^{-1} \exp(-C(\|u_0^1\|_{B_{2,\infty}^{\frac{N}{2}-1}} + \|u_0^2\|_{B_{2,\infty}^{\frac{N}{2}-1}})), \end{aligned} \tag{8}$$

then it exists a global solution (ρ, u) of the system (4) written under the form: $\rho = \rho^1 + h = \bar{\rho} + q^1 + h$ and $u = -\mu \nabla \ln \rho^1 + u^2 + u^3$ with $\partial_t \rho^1 - \mu \Delta \rho^1 = 0$, $\rho^1(0, \cdot) = \rho_0^1$ and such that:

$$h \in \tilde{C}(\mathbb{R}^+, \tilde{B}_{2,1}^{\frac{N}{2}-1, \frac{N}{2}}) \cap L^1(\mathbb{R}^+, \tilde{B}_{2,1}^{\frac{N}{2}+1, \frac{N}{2}}), \quad u^2 \in \tilde{C}(\mathbb{R}^+; B_{2,1}^{\frac{N}{2}-1}) \cap L^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}+1}),$$

and

$$u^3 \in \tilde{C}(\mathbb{R}^+; B_{2,1}^{\frac{N}{2}-1}) \cap L^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}+1}).$$

Remark 1. We would like to emphasize on the fact that the density consists in the sum of a regular function ρ_1 and of a small perturbation h verifying a transport equation. This point is very surprising in the sense that the density is governed by a hyperbolic equation which means that a priori we do not wait for any regularizing effects on the density. It seems that there is a singular behavior around the *quasi-solution* $(\rho^1, -\mu \nabla \ln \rho^1)$.

Remark 2. Up our knowledge, it is the first result of global strong solution for compressible Navier–Stokes equations with large initial data for the scaling of the equations both on the rotational and the irrotational part of the velocity.

Remark 3. We would like shortly discuss about the condition (8) which is an extension of the condition (2) and which seems quite complicated (indeed it mixed in the convection term some product between $e^{t\mu\Delta} \rho_0^1$ and $e^{t\mu\Delta} u_0^2$). We can check that if you choose initial data as in [3], it means for example $u_{0,\varepsilon}^2 = (\partial_2 \varphi_\varepsilon(x), -\partial_1 \varphi_\varepsilon(x), 0)$ with:

$$\varphi_\varepsilon = \frac{(-\log \varepsilon)^{\frac{1}{2}}}{\varepsilon^{1-\alpha}} \cos\left(\frac{x_3}{\varepsilon}\right) \phi\left(x_1, \frac{x_2}{\varepsilon^\alpha}, x_3\right),$$

with $\alpha \in]0, 1[$ and $\phi \in S(\mathbb{R}^N)$ and $q_{0,\varepsilon}^1$ of the form $\frac{1}{\varepsilon^\beta} e^{i\frac{x_3}{\varepsilon}} f(x_1, \frac{x_2}{\varepsilon^\gamma}, x_3)$ with β and γ suitably chosen in function of α and $f \in S(\mathbb{R}^N)$ then we shall verify the condition (8). In particular we need to work with $\bar{\rho}_\varepsilon = \frac{\bar{\rho}}{\varepsilon^\beta}$ in order to ensure the condition on the vacuum.

An other option in order to simplify the condition (8) consists in searching a solution of the form $u^2 = u_{L1}^2 + \bar{u}$ with u_{L1}^2 the solution of the linear equation:

$$\begin{cases} \partial_t u^2 + u^1 \cdot \nabla u^2 + u^2 \cdot \nabla u^1 - \mu \Delta u^2 + \nabla \Pi^2 = 0, \\ \operatorname{div} u^2 = 0, \quad (u^2)_{|t=0} = u_0^2 \end{cases}$$

and applying the same arguments than in [3].

2. Sketch of the proof of Theorem 1.1

We are giving a sketch of the proof in emphasizing on how to obtain the existence of global strong solution for the systems (5) and (6) which will imply Theorem 1.1.

2.1. Existence of global solution for the system (5)

Let us begin with proving the existence of global strong solution for the system (5) under the hypothesis of Theorem 1.1 and in particular the assumption (8). Following the scheme of the proof developed in [3] we are going to apply a fixed point in $X = \tilde{L}^\infty(B_{2,1}^{\frac{N}{2}-1}) \cap \tilde{L}^1(B_{2,1}^{\frac{N}{2}+1})$ (and not in the space E of [3] which is larger than X because in the sequel in order to prove the global existence of (h, u^3) , we shall assume that $\nabla \Pi^2$ is small in $\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})$). We can now write u^2 under the form $u^2 = e^{\mu t \Delta} u_0^2 + \bar{u}$ (with $u_L^2 = e^{\mu t \Delta} u_0^2$) and we have:

$$\begin{aligned} \bar{u} &= \int_0^t e^{\mu(t-s)\Delta} \mathbb{P}(u^1 \cdot \nabla u_L^2 + u^1 \cdot \nabla \bar{u} + u_L^2 \cdot \nabla u^1 + \bar{u} \cdot \nabla u^1 + u_L^2 \cdot \nabla u_L^2 + u_L^2 \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_L^2 + \bar{u} \cdot \nabla \bar{u})(s) \, ds, \\ &= \int_0^t e^{\mu(t-s)\Delta} \mathbb{P}((u^1 \cdot \nabla u_L^2 + u_L^2 \cdot \nabla u^1 + u_L^2 \cdot \nabla u_L^2) + \bar{u} \cdot \nabla(u^1 + u_L^2) + (u^1 + u_L^2) \cdot \nabla \bar{u} + \bar{u} \cdot \nabla \bar{u})(s) \, ds. \end{aligned}$$

Let us denote by Q the bilinear operator:

$$Q(u, v)(t) = \frac{1}{2} \int_0^t e^{\mu(t-s)\Delta} \mathbb{P}(u \cdot \nabla v + v \nabla u)(s) \, ds.$$

We have then:

$$\bar{u} = 2Q(u^1, u_L^2) + Q(u_L^2, u_L^2) + 2Q(\bar{u}, u^1 + u_L^2) + Q(\bar{u}, \bar{u}).$$

To do this, we shall use a standard scheme. We smooth out the data and get a sequence of local solutions $(u_n^2)_{n \in \mathbb{N}}$ on $[0, T_n]$ to (5). Next we prove uniform estimates on $(\bar{u}_n)_{n \in \mathbb{N}}$, and we have for all $n \geq 1$:

$$\begin{aligned} \|\bar{u}_n\|_X &\leq C \left(\|u^1 \cdot \nabla u_L^2 + u_L^2 \cdot \nabla u^1 + u_L^2 \cdot \nabla u_L^2\|_{\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} + \|u^1 + u_L^2\|_{\tilde{L}^1(B_{2,\infty}^{\frac{N}{2}+1})} \|\bar{u}_n\|_{\tilde{L}^\infty(B_{2,1}^{\frac{N}{2}-1})} \right) \\ &\quad + \varepsilon \|\bar{u}_n\|_{\tilde{L}^1(B_{2,1}^{\frac{N}{2}+1})} \|u^1 + u_L^2\|_{\tilde{L}^\infty(B_{2,\infty}^{\frac{N}{2}-1})} + \|\bar{u}_n\|_X^2. \end{aligned}$$

By Gronwall lemma and classical bootstrap argument we show that \bar{u}_n is globally defined under the following smallness hypothesis:

$$\|u^1 \cdot \nabla u_L^2 + u_L^2 \cdot \nabla u^1 + u_L^2 \cdot \nabla u_L^2\|_{\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} \leq C \exp(-C \|u^1 + u_L^2\|_{\tilde{L}^1(B_{2,\infty}^{\frac{N}{2}+1})}).$$

By using classical estimates in Besov spaces for the heat equation and the fact that $u^1 = -\mu \nabla \ln(1+q^1)$ with $0 \leq c \leq q_0^1 \leq M$ and q^1 verifies a heat equation, we have:

$$\|u^1 + u_L^2\|_{\tilde{L}^1(B_{2,\infty}^{\frac{N}{2}+1})} \leq C \left(\|q_0^1\|_{B_{2,\infty}^{\frac{N}{2}}} + \|u_0^2\|_{B_{2,\infty}^{\frac{N}{2}-1}} \right).$$

It is exactly the smallness hypothesis (8) and it concludes the global existence for the systems (5).

2.2. Existence of global solution for the system (6)

Let us now prove the global existence of strong solution for the system (6). To do this, we shall use a standard scheme and apply to this system the ideas developed in [5]. We smooth out the data and get a sequence of local solutions $(h_n, u_n^3)_{n \in \mathbb{N}}$ on $[0, T_n]$ to (6) by using the result of [6]. Next we prove uniform estimates on (h_n, v_n^3) (where v_n^3 is the effective velocity associated to u_n^3 , we refer to [5] for a definition) in high frequencies and on (h_n, u_n^3) in low frequencies on $[0, T_n]$ and we deduce that $T_n = +\infty$. Finally we use compactness to prove that the sequence (h^n, u_n^3) converges, up to extraction, to a solution of (5). The uniqueness is standard (see [6]).

2.3. Uniform bounds

In the sequel we are just going to explain how to get uniform estimates on (h_n, u_n^3) in suitable Besov space what is the heart of the proof. The rest is standard and follow the same lines than in [5]. For the sake of simplicity we are going to prove uniform estimate on (h, u^3) (we forget here the index n of the sequel). We start with recalling some estimates in Besov spaces for the linear system associated to the system (6) that we can write under the following form:

$$\begin{cases} \partial_t h + v_1 \cdot \nabla h + \operatorname{div} u^3 = F, \\ \partial_t u^3 + v_1 \cdot \nabla u^3 - \mu \Delta u^3 + a \nabla h = G, \\ (h, u^3)|_{t=0} = (h_0, u_0^3). \end{cases} \tag{9}$$

Here (F, G) are external forces and v_1 a vector field with a regularity that we shall precise in Proposition 2.1 which has been proved in [1,5] by using two different method. In particular in [5], we introduce the notion of *effective velocity*.

Proposition 2.1. *Let $p \leq \max(4, N)$. Let $s = \frac{N}{p}$ and $s' = \frac{N}{2} - 1$. Let (h, u^3) the solution of (9). There exists a constant C depending only on μ, N, s and s' such that the following estimate holds:*

$$\begin{aligned} \|(h^2, u^2)(t)\|_{\tilde{B}_{2,p,1}^{s'-1,s} \times \tilde{B}_{2,p,1}^{s'-1,s-1}} &+ \int_0^t \|(h^2, u^2)(s)\|_{\tilde{B}_{2,p,1}^{s'+1,s} \times \tilde{B}_{2,p,1}^{s'+1,s+1}} \, ds \\ &\leq C \left(\|(h_0^2, u_0^2)\|_{\tilde{B}_{2,p,1}^{s'-1,s} \times \tilde{B}_{2,p,1}^{s'-1,s-1}} + \int_0^t e^{-V(s)} \|(F, G)(s)\|_{\tilde{B}_{2,p,1}^{s'-1,s} \times \tilde{B}_{2,p,1}^{s'-1,s-1}} \, ds \right) \end{aligned}$$

with $V(T) = \int_0^T \|\nabla v_1(s)\|_{L^\infty} ds$.

We now are going to apply this proposition to the system (9) with $v_1 = u$ and

$$F = -v \cdot \nabla q^1 - h \operatorname{div} u^3 - q^1 \operatorname{div} u^3,$$

$$G = -a \nabla q^1 + \nabla \Pi^2 - u^3 \cdot \nabla w + \mu \nabla \ln \rho \cdot Dv + \mu \nabla (\ln \rho - \ln \rho^1) \cdot Du^1.$$

In order to obtain uniform estimates on (h, u^3) , it is only a matter of proving appropriate estimates on (F, G) by using properties of continuity on the paraproduct (we refer in particular to the appendix of [5] when the Besov spaces are hybrid). The only technical point is to ensure that we can use a bootstrap argument via the Gronwall lemma in order to estimate (h, u^3) in X_3 with:

$$X_{3,T} = (\tilde{L}_T^\infty(\tilde{B}_{2,1}^{\frac{N}{2}-1, \frac{N}{2}}) \cap L_T^1(\tilde{B}_{2,1}^{\frac{N}{2}+1, \frac{N}{2}})) \times (\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}-1}) \cap L_T^1(B_{2,1}^{\frac{N}{2}+1})).$$

In particular in order to obtain global solution for the solution (h, u^3) we need some smallness hypothesis in $L^1(B_{2,1}^{\frac{N}{2}-1})$ on the following terms ∇q^1 , $\nabla \Pi^2$, $\nabla \ln \rho^1 \cdot Du^2$ and on $u^2 \cdot \nabla q^1$ in $\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1, \frac{N}{2}})$. This smallness hypothesis is ensured by the hypothesis (8) and (7), in particular in order to deal with the term $a \nabla q^1$ it suffices to observe that this term is sufficiently small in $L^1(B_{2,1}^{\frac{N}{2}-1})$ via the hypothesis (7) with l large enough and the fact that q^1 verifies a heat equation (see for more details [9]). We can deal easily with all the previous terms via the condition (8) except for $\nabla \Pi^2$ where it is not so clear. However by applying the operator div to the system (5), we obtain:

$$\Delta \Pi^2 = -\operatorname{div}(u^2 \cdot \nabla u^1 + u^1 \cdot \nabla u^2 + u^2 \cdot \nabla u^2).$$

It suffices now to use elliptic estimates and hypothesis (8) in order to ensure smallness hypothesis on $\nabla \Pi^2$ in order to use a classical bootstrap argument. From a standard bootstrap argument (see [5]) which consists in using Gronwall lemma in order to deal with the linear term in (h, u^3) in F and G we obtain the following estimate:

$$\begin{aligned} \|(h, u^3)\|_{X_{3,T}} \leq & C e^{V(T)} (\|h_0\|_{\tilde{B}_{2,1}^{\frac{N}{2}-1, \frac{N}{2}}} + \|u_0^3\|_{B_{2,1}^{\frac{N}{2}-1}} + \|\nabla q^1\|_{\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} + \|u^2 \cdot \nabla q^1\|_{\tilde{L}^1(\tilde{B}_{2,1}^{\frac{N}{2}-1, \frac{N}{2}})} \\ & + \|\nabla \Pi^2\|_{\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} + \|\nabla \ln \rho^1 \cdot Du^2\|_{\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} + \|u^3 \cdot \nabla u^3\|_{\tilde{L}^1(B_{2,1}^{\frac{N}{2}-1})}). \end{aligned}$$

The hypothesis (7) and (8) ensures enough smallness in order to use a bootstrap argument what concludes the proof of Theorem 1.1.

References

- [1] F. Charve, R. Danchin, A global existence result for the compressible Navier–Stokes equations in the critical L^p framework, *Archive for Rational Mechanics and Analysis* 198 (1) (2010) 233–271.
- [2] F. Charve, B. Haspot, Existence of strong solutions in larger space for the shallow-water system, *Advances in Differential Equations*, submitted for publication.
- [3] J.-Y. Chemin, I. Gallagher, Wellposedness and stability results for the Navier–Stokes equations in \mathbb{R}^3 , *Annales de l’Institut H. Poincaré, Analyse non Linéaire* 26 (2) (2009) 599–624.
- [4] J.-Y. Chemin, I. Gallagher, M. Paicu, Global regularity for some classes of large solutions to the Navier–Stokes equations, *Annals of Mathematics* 173 (2) (2011) 983–1012.
- [5] B. Haspot, Existence of global strong solutions in critical spaces for barotropic viscous fluids, *Archive for Rational Mechanics and Analysis* 202 (2) (2011) 427–460.
- [6] B. Haspot, Existence of strong solutions in critical spaces for barotropic viscous fluids in larger spaces, *Journal of Differential Equations* 251 (8) (2011) 2262–2295.
- [7] B. Haspot, Existence of strong global solutions for the shallow-water equations with large initial data, preprint, arXiv:1110.6100.
- [8] B. Haspot, Existence of global strong solutions for the Saint-Venant system with large initial data on the irrotational part of the velocity, *C. R. Acad. Sci. Paris, Ser. I* 350 (5–6) (2012) 249–254.
- [9] B. Haspot, Global existence of strong solution for shallow water system with large initial data on the irrotational part, preprint, arXiv:1201.5456.