



Algebra/Algebraic Geometry

## Projective geometry for blueprints

*Geometrie projective pour les canevas bleus*Javier López Peña<sup>a,1</sup>, Oliver Lorscheid<sup>b</sup><sup>a</sup> Department of Mathematics, University College London, 25 Gower Street, London WC1E 6BT, United Kingdom<sup>b</sup> Department of Mathematics, University of Wuppertal, Gaußstr. 20, 42097 Wuppertal, Germany

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## ABSTRACT

In this Note, we generalize the Proj-construction from usual schemes to blue schemes. This yields the definition of projective space and projective varieties over a blueprint. In particular, it is possible to descend closed subvarieties of a projective space to a canonical  $\mathbb{F}_1$ -model. We discuss this in case of the Grassmannian  $\text{Gr}(2, 4)$ .

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## R É S U M É

Dans cette Note, nous généralisons la Proj-construction des schémas usuels aux schémas bleus. Cela entraîne la définition d'espace projectif et de variétés projectives sur un canevas bleu. En particulier, il est possible de descendre une sous-variété fermée d'un espace projectif en un  $\mathbb{F}_1$ -modèle canonique. Nous discutons cela dans le cas de la Grassmannienne  $\text{Gr}(2, 4)$ .

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## 1. Introduction

Blueprints are a common generalization of commutative (semi)rings and monoids. The associated geometric objects, blue schemes, are therefore a common generalization of usual scheme theory and  $\mathbb{F}_1$ -geometry (as considered by Kato [5], Deitmar [3] and Connes–Consani [2]). The possibility of forming semiring schemes allows us to talk about idempotent schemes and tropical schemes (cf. [11]). All this is worked out in [9].

It is known, though not covered in literature yet, that the Proj-construction from usual algebraic geometry has an analogue in  $\mathbb{F}_1$ -geometry (after Kato, Deitmar and Connes–Consani). In this note we describe a generalization of this to blueprints. Privately, Koen Thas has announced a treatment of Proj for monoidal schemes (see [13]).

We follow the notations and conventions of [10]. Namely, all blueprints that appear in this note are proper and with a zero. We remark that the following constructions can be carried out for the more general notion of a blueprint as considered in [9]; the reason that we restrict to proper blueprints with a zero is that this allows us to adopt a notation that is common in  $\mathbb{F}_1$ -geometry.

Namely, we denote by  $\mathbb{A}_B^n$  the (blue) affine  $n$ -space  $\text{Spec}(B[T_1, \dots, T_n])$  over a blueprint  $B$ . In case of a ring, this does not equal the usual affine  $n$ -space since  $B[T_1, \dots, T_n]$  is not closed under addition. Therefore, we denote the usual affine

E-mail addresses: [jlp@math.ucl.ac.uk](mailto:jlp@math.ucl.ac.uk) (J. López Peña), [lorscheid@math.uni-wuppertal.de](mailto:lorscheid@math.uni-wuppertal.de) (O. Lorscheid).<sup>1</sup> J. López Peña's research was supported by MCIM grant MTM2010-20940-C02-01, research group FQM-266 (Junta de Andalucía) and Max-Planck Institute for Mathematics in Bonn.

$n$ -space over a ring  $B$  by  ${}^+ \mathbb{A}_B^n = \text{Spec}(B[T_1, \dots, T_n]^+)$ . Similarly, we use a superscript “+” for the usual projective space  ${}^+ \mathbb{P}_B^n$  and the usual Grassmannian  $\text{Gr}(k, n)_B^+$  over a ring  $B$ .

**2. Graded blueprints and Proj**

Let  $B$  be a blueprint and  $M$  a subset of  $B$ . We say that  $M$  is *additively closed* in  $B$  if for all additive relations  $b \equiv \sum a_i$  with  $a_i \in M$  also  $b$  is an element of  $M$ . Note that, in particular,  $0$  is an element of  $M$ . A *graded blueprint* is a blueprint  $B$  together with additively closed subsets  $B_i$  for  $i \in \mathbb{N}$  such that  $1 \in B_0$ , such that for all  $i, j \in \mathbb{N}$  and  $a \in B_i, b \in B_j$ , the product  $ab$  is an element of  $B_{i+j}$  and such that for every  $b \in B$ , there are a unique finite subset  $I$  of  $\mathbb{N}$  and unique non-zero elements  $a_i \in B_i$  for every  $i \in I$  such that  $b = \sum a_i$ . An element of  $\bigcup_{i \geq 0} B_i$  is called *homogeneous*. If  $a \in B_i$  is non-zero, then we say, more specifically, that  $a$  is *homogeneous of degree  $i$* .

We collect some immediate facts for a graded blueprint  $B$  as above. The subset  $B_0$  is multiplicatively closed, i.e.  $B_0$  can be seen as a subblueprint of  $B$ . The subblueprint  $B_0$  equals  $B$  if and only if for all  $i > 0, B_i = \{0\}$ . In this case we say that  $B$  is *trivially graded*. By the uniqueness of the decomposition into homogeneous elements, we have  $B_i \cap B_j = \{0\}$  for  $i \neq j$ . This means that the union  $\bigcup_{i \geq 0} B_i$  has the structure of a wedge product  $\bigvee_{i \geq 0} B_i$ . Since  $\bigvee_{i \geq 0} B_i$  is multiplicatively closed, it can be seen as a subblueprint of  $B$ . We define  $B_{\text{hom}} = \bigvee_{i \geq 0} B_i$  and call the subblueprint  $B_{\text{hom}}$  the *homogeneous part of  $B$* .

Let  $S$  be a multiplicative subset of  $B$ . If  $b/s$  is an element of the localization  $S^{-1}B$  where  $f$  is homogeneous of degree  $i$  and  $s$  is homogeneous of degree  $j$ , then we say that  $b/s$  is a homogeneous element of degree  $i - j$ . We define  $S^{-1}B_0$  as the subset of homogeneous elements of degree  $0$ . It is multiplicatively closed, and inherits thus a subblueprint structure from  $S^{-1}B$ . If  $S$  is the complement of a prime ideal  $\mathfrak{p}$ , then we write  $B_{(\mathfrak{p})}$  for the subblueprint  $(B_{\mathfrak{p}})_0$  of homogeneous elements of degree  $0$  in  $B_{\mathfrak{p}}$ .

An ideal  $I$  of a graded blueprint  $B$  is called *homogeneous* if it is generated by homogeneous elements, i.e. if for every  $c \in I$ , there are homogeneous elements  $p_i, q_j \in I$  and elements  $a_i, b_j \in B$  and an additive relation  $\sum a_i p_i + c = \sum b_j q_j$  in  $B$ .

Let  $B$  be a graded blueprint. Then we define  $\text{Proj } B$  as the set of all homogeneous prime ideals  $\mathfrak{p}$  of  $B$  that do not contain  $B_{\text{hom}}^+ = \bigvee_{i > 0} B_i$ . The set  $X = \text{Proj } B$  comes together with the topology that is defined by the basis

$$U_h = \{\mathfrak{p} \in X \mid h \notin \mathfrak{p}\}$$

where  $h$  ranges through  $B_{\text{hom}}$  and with a structure sheaf  $\mathcal{O}_X$  that is the sheafification of the association  $U_h \mapsto B[h^{-1}]_0$  where  $B[h^{-1}]$  is the localization of  $B$  at  $S = \{h^i\}_{i \geq 0}$ .

Note that if  $B$  is a ring, the above definitions yield the usual construction of  $\text{Proj } B$  for graded rings. In complete analogy to the case of graded rings, one proves the following theorem:

**Theorem 1.** *The space  $X = \text{Proj } B$  together with  $\mathcal{O}_X$  is a blue scheme. The stalk at a point  $\mathfrak{p} \in \text{Proj } B$  is  $\mathcal{O}_{X, \mathfrak{p}} = B_{(\mathfrak{p})}$ . If  $h \in B_{\text{hom}}^+$ , then  $U_h \simeq \text{Spec } B[h^{-1}]_0$ . The inclusions  $B_0 \hookrightarrow B[h^{-1}]_0$  yield morphisms  $\text{Spec } B[h^{-1}]_0 \rightarrow \text{Spec } B_0$ , which glue to a structural morphism  $\text{Proj } B \rightarrow \text{Spec } B_0$ .  $\square$*

If  $B$  is a graded blueprint, then the associated semiring  $B^+$  inherits a grading. Namely, let  $B_{\text{hom}} = \bigvee_{i \geq 0} B_i$  the homogeneous part of  $B$ . Then we can define  $B_i^+$  as the additive closure of  $B_i$  in  $B^+$ , i.e. as the set of all  $b \in B$  such that there is an additive relation of the form  $b \equiv \sum a_k$  in  $B$  with  $a_k \in B_i$ . Then  $\bigvee B_i^+$  defines a grading of  $B^+$ . Similarly, the grading of  $B$  induces a grading on a tensor product  $B \otimes_C D$  with respect to blueprint morphisms  $C \rightarrow B$  and  $C \rightarrow D$  under the assumption that the image of  $C \rightarrow B$  is contained in  $B_0$ . Consequently, a grading of  $B$  implies a grading of  $B_{\text{inv}} = B \otimes_{\mathbb{F}_1} \mathbb{F}_{12}$  (see [9, Lemma 1.4] and [10, p. 11]) and of the ring  $B_{\mathbb{Z}}^+ = B_{\text{inv}}^+$ . Analogously, if both  $B$  and  $D$  are graded and the image of  $C$  lies in both  $B_0$  and  $D_0$ , then  $B \otimes_C D$  inherits a grading from the gradings of  $B$  and  $D$ .

**3. Projective space**

The functor  $\text{Proj}$  allows the definition of the projective space  $\mathbb{P}_B^n$  over a blueprint  $B$ . Namely, the free blueprint  $C = B[T_0, \dots, T_n]$  over  $B$  comes together with a natural grading (cf. [9, Section 1.12] for the definition of free blueprints). Namely,  $C_i$  consists of all monomials  $bT_0^{e_0} \dots T_n^{e_n}$  such that  $e_0 + \dots + e_n = i$  where  $b \in B$ . Note that  $C_0 = B$  and  $C_{\text{hom}} = C$ . The projective space  $\mathbb{P}_B^n$  is defined as  $\text{Proj } B[T_0, \dots, T_n]$ . It comes together with a structure morphism  $\mathbb{P}_B^n \rightarrow \text{Spec } B$ .

In case of  $B = \mathbb{F}_1$ , the projective space  $\mathbb{P}_{\mathbb{F}_1}^n$  is the monoidal scheme that is known from  $\mathbb{F}_1$ -geometry (see [4], [1, Section 3.1.4]) and [10, Ex. 1.6]). The topological space of  $\mathbb{P}_{\mathbb{F}_1}^n$  is finite. Its points correspond to the homogeneous prime ideals  $(S_i)_{i \in I}$  of  $\mathbb{F}_1[S_0, \dots, S_n]$  where  $I$  ranges through all proper subsets of  $\{0, \dots, n\}$ .

In case of a ring  $B$ , the projective space  $\mathbb{P}_B^n$  does not coincide with the usual projective space since the free blueprint  $B[S_0, \dots, S_n]$  is not a ring, but merely the blueprint of all monomials of the form  $bS_0^{e_0} \dots S_n^{e_n}$  with  $b \in B$ . However, the associated scheme  ${}^+ \mathbb{P}_B^n = (\mathbb{P}_B^n)^+$  coincides with the usual projective space over  $B$ , which equals  $\text{Proj } B[S_0, \dots, S_n]^+$ .

**4. Closed subschemes**

Let  $\mathcal{X}$  be a scheme of finite type. By an  $\mathbb{F}_1$ -model of  $\mathcal{X}$  we mean a blue scheme  $X$  of finite type such that  $X_{\mathbb{Z}}^+$  is isomorphic to  $\mathcal{X}$ . Since a finitely generated  $\mathbb{Z}$ -algebra is, by definition, generated by a finitely generated multiplicative

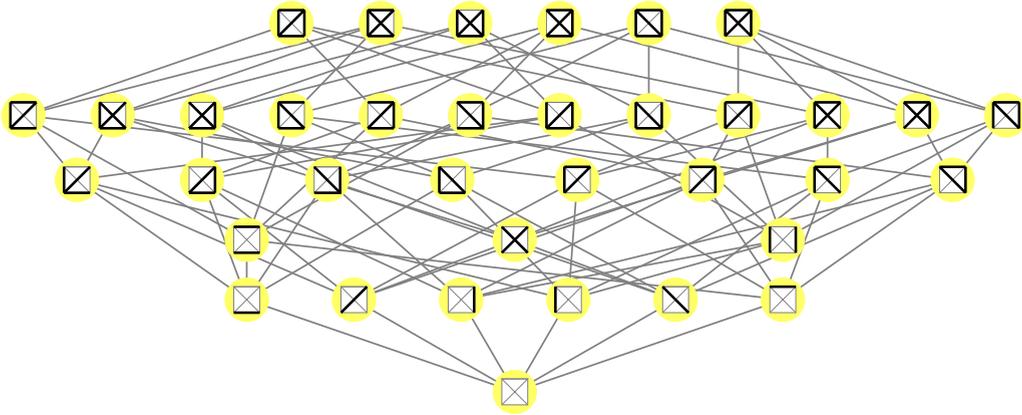


Fig. 1. Points of the Grassmannian  $\text{Gr}(2, 4)_{\mathbb{F}_1}$ . Generator  $x_{ij}$  belonging to an ideal is depicted as segment  $i-j$  in  $\begin{matrix} 4 \\ 1 \end{matrix} \begin{matrix} \text{X} \\ \text{X} \\ \text{X} \\ \text{X} \\ \text{X} \end{matrix} \begin{matrix} 3 \\ 2 \end{matrix}$ .

subset as a  $\mathbb{Z}$ -module, every scheme of finite type has an  $\mathbb{F}_1$ -model. It is, on the contrary, true that a scheme of finite type possesses a large number of  $\mathbb{F}_1$ -models.

Given a scheme  $\mathcal{X}$  with an  $\mathbb{F}_1$ -model  $X$ , we can associate to every closed subscheme  $\mathcal{Y}$  of  $\mathcal{X}$  the following closed subscheme  $Y$  of  $X$ , which is an  $\mathbb{F}_1$ -model of  $\mathcal{Y}$ . In case that  $X = \text{Spec } B$  is the spectrum of a blueprint  $B = A//\mathcal{R}$ , and thus  $\mathcal{X} \simeq \text{Spec } B_{\mathbb{Z}}^+$  is an affine scheme, we can define  $Y$  as  $\text{Spec } C$  for  $C = A//\mathcal{R}(Y)$  where  $\mathcal{R}(Y)$  is the pre-addition that contains  $\sum a_i \equiv \sum b_j$  whenever  $\sum a_i = \sum b_j$  holds in the coordinate ring  $\Gamma \mathcal{Y}$  of  $\mathcal{Y}$ . This is a process that we used already in [10, Section 3].

Since localizations commute with additive closures, i.e.  $(S^{-1}B)_{\mathbb{Z}}^+ = S^{-1}(B_{\mathbb{Z}}^+)$  where  $S$  is a multiplicative subset of  $B$ , the above process is compatible with the restriction to affine opens  $U \subset X$ . This means that given  $U = \text{Spec}(S^{-1}B)$ , which is an  $\mathbb{F}_1$ -model for  $\mathcal{X}' = U_{\mathbb{Z}}^+$ , then the  $\mathbb{F}_1$ -model  $Y'$  that is associated to the closed subscheme  $\mathcal{Y}' = \mathcal{X}' \times_{\mathcal{X}} \mathcal{Y}$  of  $\mathcal{X}'$  by the above process is the spectrum of the blueprint  $S^{-1}C$ . Consequently, we can associate with every closed subscheme  $\mathcal{Y}$  of a scheme  $\mathcal{X}$  with an  $\mathbb{F}_1$ -model  $X$  a closed subscheme  $Y$  of  $X$ , which is an  $\mathbb{F}_1$ -model of  $\mathcal{Y}$ ; namely, we apply the above process to all affine open subschemes of  $\mathcal{X}$  and glue them together, which is possible since additive closures commute with localizations.

In case of a projective variety, i.e. a closed subscheme  $\mathcal{Y}$  of a projective space  ${}^+ \mathbb{P}_{\mathbb{Z}}^n$ , we derive the following description of the associated  $\mathbb{F}_1$ -model  $Y$  in  $\mathbb{P}_{\mathbb{F}_1}^n$  by homogeneous coordinate rings. Let  $C$  be the homogeneous coordinate ring of  $\mathcal{Y}$ , which is a quotient of  $\mathbb{Z}[S_0, \dots, S_n]^+$  by a homogeneous ideal  $I$ . Let  $\mathcal{R}$  be the pre-addition on  $\mathbb{F}_1[S_0, \dots, S_n]$  that consists of all relations  $\sum a_i \equiv \sum b_j$  such that  $\sum a_i = \sum b_j$  in  $C$ . Then  $B = \mathbb{F}_1[S_0, \dots, S_n]//\mathcal{R}$  inherits a grading from  $\mathbb{F}_1[S_0, \dots, S_n]$  by defining  $B_i$  as the image of  $\mathbb{F}_1[S_0, \dots, S_n]_i$  in  $B$ . Note that  $B \subset C$  and that the sets  $B_i$  equal the intersections  $B_i = C_i \cap B$  for  $i \geq 0$  where  $C_i$  is the homogeneous part of degree  $i$  of  $C$ . Then the  $\mathbb{F}_1$ -model  $Y$  of  $\mathcal{Y}$  equals  $\text{Proj } B$ .

### 5. $\mathbb{F}_1$ -models for Grassmannians

One of the simplest examples of projective varieties that is not toric is the Grassmannian  $\text{Gr}(2, 4)$ . The problem of finding  $\mathbb{F}_1$ -models for Grassmannians was originally posed by Soulé in [12], and solved by the authors by obtaining a torification from the Schubert cell decomposition (cf. [8,7]). In this note, we present  $\mathbb{F}_1$ -models for Grassmannians as projective varieties defined through (homogeneous) blueprints. The proposed construction fits within a more general framework for obtaining blueprints and totally positive blueprints from cluster data (cf. [6]).

Classically, the coordinate ring for  $\text{Gr}(k, n)$  is obtained by quotienting out the homogeneous coordinate ring of the projective space  $\mathbb{P}^{\binom{n}{k}-1}$  by the homogeneous ideal generated by the Plücker relations. A similar construction can be carried out using the framework of (graded) blueprints. We make that construction explicit for  $\text{Gr}(2, 4)$ .

Define the blueprint  $\mathcal{O}_{\mathbb{F}_1}(\text{Gr}(2, 4)) = \mathbb{F}_1[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]//\mathcal{R}$  where the congruence  $\mathcal{R}$  is generated by the Plücker relation  $x_{12}x_{34} + x_{14}x_{23} \equiv x_{13}x_{24}$  (the signs have been picked to ensure that the totally positive part of the Grassmannian is preserved, cf. [6]). Since  $\mathcal{R}$  is generated by a homogeneous relation,  $\mathcal{O}_{\mathbb{F}_1}(\text{Gr}(2, 4))$  inherits a grading from the canonical morphism

$$\pi : \mathbb{F}_1[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}] \longrightarrow \mathbb{F}_1[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]//\mathcal{R}.$$

Let  $\text{Gr}(2, 4)_{\mathbb{F}_1} := \text{Proj}(\mathcal{O}_{\mathbb{F}_1}(\text{Gr}(2, 4)))$ . The base extension  $\text{Gr}(2, 4)_{\mathbb{Z}}^+$  is the usual Grassmannian, and  $\pi$  defines a closed embedding of  $\text{Gr}(2, 4)_{\mathbb{F}_1}$  into  $\mathbb{P}_{\mathbb{F}_1}^5$ , which extends to the classical Plücker embedding  $\text{Gr}(2, 4)_{\mathbb{Z}}^+ \hookrightarrow \mathbb{P}_{\mathbb{Z}}^5$ .

Homogeneous prime ideals in  $\mathcal{O}_{\mathbb{F}_1}(\text{Gr}(2, 4))$  are described by their generators as the proper subsets  $I \subsetneq \{x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}\}$  such that  $I$  is either contained in one of the sets  $\{x_{12}, x_{34}\}$ ,  $\{x_{14}, x_{23}\}$ ,  $\{x_{13}, x_{24}\}$ , or otherwise  $I$  has a non-empty intersection with all three of them. In other words,  $I$  cannot contain elements in two of the above sets without also containing an element of the third one.  $\text{Gr}(2, 4)_{\mathbb{F}_1}$  is depicted in Fig. 1. It consists of  $6 + 12 + 11 + 6 + 1 = 36$  prime ideals of ranks

0, 1, 2, 3 and 4, respectively (cf. [10, Def. 2.3] for the definition of rank), thus resulting in a model essentially different to the one presented in [8], which had 35 points corresponding to the coefficients of  $N_{\text{Gr}(2,4)}(q) = 6 + 12(q-1) + 11(q-1)^2 + 5(q-1)^3 + 1(q-1)^4$ . In spite of arising from different constructions, both  $\mathbb{F}_1$ -models for  $\text{Gr}(2,4)$  have  $6 = \binom{4}{2}$  closed points, supporting the naive combinatorial interpretation of  $\text{Gr}(2,4)_{\mathbb{F}_1}$ . These six points correspond to the  $\mathbb{F}_1$ -rational Tits points of  $\text{Gr}(2,4)_{\mathbb{F}_1}$ , which reflect the naive notion of  $\mathbb{F}_1$ -rational points of an  $\mathbb{F}_1$ -scheme (cf. [10, Section 2.2]).

As in the classical setting, the Grassmannian  $\text{Gr}(2,4)_{\mathbb{F}_1}$  does admit a covering by six  $\mathbb{F}_1$ -models of affine 4-space, which correspond to the open subsets of  $\text{Gr}(2,4)_{\mathbb{F}_1}$  where one of the generators is non-zero. However, these  $\mathbb{F}_1$ -models of affine 4-space are not the standard model  $\mathbb{A}_{\mathbb{F}_1}^4 = \text{Spec}(\mathbb{F}_1[a, b, c, d])$ , but the “ $2 \times 2$ -matrices”  $M_{2,\mathbb{F}_1} = \text{Spec}(\mathbb{F}_1[a, b, c, d, D]// (ad \equiv bc + D))$  in case that one of  $x_{12}$ ,  $x_{34}$ ,  $x_{14}$  or  $x_{23}$  is non-zero, and the “twisted  $2 \times 2$ -matrices”  $M_{2,\mathbb{F}_1}^{\tau} = \text{Spec}(\mathbb{F}_1[a, b, c, d, D]// (ad + bc \equiv D))$  in case that one of  $x_{13}$  or  $x_{24}$  is non-zero.

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