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Algebra

On the cardinality of stable star operations of finite type on an integral domain

Sur le cardinal des opérations étoile stables de type fini d'un anneau intègre

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ABSTRACT

Let D be an integral domain and $SF_s(D)$ be the set of stable star operations of finite type on D. In this note, we show that if Ω is the set of nonzero prime ideals P of D with $P^t = D$, then $|\Omega| + 1 \leqslant |SF_s(D)| \leqslant 2^{|\Omega|}$. We also show that if $|\Omega| < \infty$, then $|SF_s(D)| = |\Omega| + 1$ if and only if Ω is linearly ordered under inclusion; and $|SF_s(D)| = 2^{|\Omega|}$ if and only if each pair of elements in Ω are incomparable.

RÉSUMÉ

Soit D un anneau intègre et $SF_s(D)$ l'ensemble des opérations étoile, stables, de type fini sur D. Nous montrons dans cette note que, si Ω désigne l'ensemble des idéaux premiers non nuls P de D tels que $P^t = D$, alors $|\Omega| + 1 \leqslant |SF_s(D)| \leqslant 2^{|\Omega|}$. Nous montrons également que, si $|\Omega| < \infty$, alors $|SF_s(D)| = |\Omega| + 1$ si et seulement si Ω est totalement ordonné par l'inclusion et $|SF_s(D)| = 2^{|\Omega|}$ si et seulement si les éléments de Ω sont deux à deux incomparables.

1. Introduction

Let D be an integral domain with quotient field K. Let $\mathbf{F}(D)$ (resp., $\mathbf{f}(D)$) be the set of nonzero fractional (resp., nonzero finitely generated fractional) ideals of D; so $\mathbf{f}(D) \subseteq \mathbf{F}(D)$. A mapping $I \mapsto I^*$ of $\mathbf{F}(D)$ into $\mathbf{F}(D)$ is called a *star operation* on D if for all $0 \neq a \in K$ and $I, J \in \mathbf{F}(D)$, the following conditions are satisfied:

(1) $(aD)^* = aD$ and $(aI)^* = aI^*$,
(2) $I \subseteq I^*$; $I \subseteq J$ implies $I^* \subseteq J^*$, and
(3) $(I^*)^* = I^*$.

Given any star operation $*$ on D, one can construct two new star operations $*_f$ and $*_w$ on D. The $*_f$-operation is defined by $I^{*_f} = \bigcup\{J^* \mid J \subseteq I$ and $J \in \mathbf{f}(D)\}$ and the $*_w$-operation is defined by $I^{*_w} = \{x \in K \mid xJ \subseteq I$ for some $J \in \mathbf{f}(D)$ with $J^* = D\}$. Obviously, $(*_f)_f = *_f$ and $(*_f)_w = (*_w)_f = *_w$.

A star operation $*$ on D is said to be of *finite type* if $*_f = *$. An $I \in \mathbf{F}(D)$ is called a *$*$-ideal* if $I^* = I$, while a $*$-ideal is a *maximal $*$-ideal* if it is maximal among proper integral $*$-ideals of D. Let $*$-Max(D) denote the set of maximal $*$-ideals

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of D. It is well known that a maximal $*_f$-ideal is a prime ideal, each prime ideal minimal over a $*_f$-ideal is a $*_f$-ideal, and $*_f$-Max$(D) \neq \emptyset$ when D is not a field. The most well-known examples of star operations are the d-, v-, t-, and w-operations. The d-operation is just the identity function on $\mathbf{F}(D)$, i.e., $I^d = I$ for all $I \in \mathbf{F}(D)$; so $d = d_f = d_w$. The v-operation is defined by $I^v = (I^{-1})^{-1}$, where $I^{-1} = \{x \in K \mid xI \subseteq D\}$, while the t-operation (resp., w-operation) is given by $t = v_f$ (resp., $w = v_w$). For two star operations $*$ and $*_1$ on D, we mean by $* \leqslant *_1$ that $I^* \subseteq I^{*_1}$ for all $I \in \mathbf{F}(D)$. Clearly, if $* \leqslant *_1$, then $*_f \leqslant (*_1)_f$, $*_w \leqslant (*_1)_w$, and $*_w \leqslant *_f \leqslant *$. We know that if $*$ is any star operation on D, then $d \leqslant * \leqslant v$, and hence $d \leqslant *_f \leqslant t$ and $d \leqslant *_w \leqslant w$. For basic properties of star operations, see [5, Sections 32 and 34].

A star operation $*$ on D is said to be *stable* if $(I \cap J)^* = I^* \cap J^*$ for each $I, J \in \mathbf{F}(D)$. The last statement of the following lemma provides a very useful characterization of stable star operations of finite type. We will use this fact without any reference in the subsequent argument.

Lemma 0. *Let $*$ be a star operation on D.*

(1) [2, Theorem 2.16] $*_f$*-Max*$(D) = *_w$*-Max*(D).
(2) [2, Corollary 2.10] $I^{*_w} = \bigcap_{P \in *_f\text{-Max}(D)} ID_P$ *for all* $I \in \mathbf{F}(D)$.
(3) *(Cf. [1, Corollary 4.2].) $*$ is stable and of finite type if and only if $* = *_w$.*

Let $SF_s(D)$ (resp., $S(D)$, $SF(D)$) be the set of stable star operations of finite type (resp., star operations, star operations of finite type) on D; so $SF_s(D) \subseteq SF(D) \subseteq S(D)$. It is clear that $|SF_s(D)| = 1$ if and only if $d = w$, if and only if every maximal ideal of D is a t-ideal [8, Proposition 2.2]. This type of integral domains is sometimes called a DW-domain and has been studied by many authors [3,4,8,9]. For example, a Prüfer domain or an integral domain of (Krull) dimension one is a DW-domain. In particular, if $|S(D)| < \infty$, then $d = w$ [6, Proposition 2.1], and thus $|SF_s(D)| = 1$. In [6], the authors studied integral domains D with $|S(D)| \leqslant 2$ in the integrally closed and Noetherian cases. Among many interesting results, they showed that if $|SF_s(D)| = 2$, then D has at most one prime ideal that is not a w-ideal [6, Corollary 2.8]. They also showed that if D is a Krull domain, then $|SF_s(D)| = 2$ if and only if $\dim(D) = 2$ and D has a unique maximal ideal of height two [6, Corollary 2.10]. It is easy to show that if D is a Krull domain, then D has a unique prime ideal that is not a w-ideal if and only if $\dim(D) = 2$ and D has a unique maximal ideal of height two. It therefore seems natural to ask if the converse of [6, Corollary 2.8] is true, which has inspired this article.

Let Ω be the set of prime ideals P of D such that $P^t = D$. In this paper, we compute $|SF_s(D)|$ for any integral domain D. Precisely, we show that $|\Omega| + 1 \leqslant |SF_s(D)| \leqslant 2^{|\Omega|}$. We also show that if $|\Omega| < \infty$, then $|SF_s(D)| = |\Omega| + 1$ if and only if Ω is linearly ordered under inclusion; and $|SF_s(D)| = 2^{|\Omega|}$ if and only if each pair of elements in Ω are incomparable. As a corollary, we have that $|SF_s(D)| = 2$ if and only if D has a unique maximal ideal that is not a w-ideal.

2. Main results

Let D be an integral domain and Ω be the set of prime ideals P of D with $P^t = D$. Let $SF_s(D)$ be the set of stable star operations of finite type on D. In this section, we show that $|\Omega| + 1 \leqslant |SF_s(D)| \leqslant 2^{|\Omega|}$.

Lemma 1. *For a nonzero prime ideal P of D, let*

$$E^{*_P} = ED_P \cap E^w$$

for all $E \in \mathbf{F}(D)$.

(1) *$*_P$ is a stable star operation of finite type.*
(2) *$P^t \subsetneq D$ if and only if $*_P = w$.*

Proof. (1) By [6, Proposition 2.7], $*_P$ is a star operation of finite type. It is also clear that $*_P$ is stable because the w-operation is stable.

(2) Assume that $P^t \subsetneq D$. Let Q be a maximal t-ideal of D with $P \subseteq Q$. Then, for each $E \in \mathbf{F}(D)$, we have $E^w \subseteq E^w D_Q = ED_Q \subseteq ED_P$, and thus $E^{*_P} = ED_P \cap E^w = E^w$. Thus $*_P = w$. For the converse, assume $P^t = D$. Then $P^{*_P} = PD_P \cap P^w = PD_P \cap D = P \subsetneq D = P^w$, and thus $*_P \neq w$. Hence $*_P = w$ implies $P^t \subsetneq D$. $\square$

Lemma 2. *For each $M_1, M_2 \in \Omega$,*

(1) *$*_{M_1} \leqslant *_{M_2}$ if and only if $M_1 \supseteq M_2$.*
(2) *$M_1 \neq M_2$ if and only if $*_{M_1} \neq *_{M_2}$.*

Proof. (1) Assume $*_{M_1} \leqslant *_{M_2}$. If $M_2 \nsubseteq M_1$, then $D = (M_2)^{*_{M_1}} \subseteq (M_2)^{*_{M_2}} = M_2$, a contradiction. Thus $M_2 \subseteq M_1$. Conversely, if $M_2 \subseteq M_1$, then $ED_{M_1} \subseteq ED_{M_2}$, and thus $E^{*_{M_1}} \subseteq E^{*_{M_2}}$ for all $E \in \mathbf{F}(D)$. Thus $*_{M_1} \leqslant *_{M_2}$.

(2) For convenience, assume that $M_2 \nsubseteq M_1$. Then $(M_1)^{*_{M_1}} = M_1 \subsetneq D = (M_2)^{*_{M_1}}$, and thus $*_{M_1} \neq *_{M_2}$. The converse is clear. □

Recall that $I^{*_w} = \bigcap_{P \in *_f\text{-Max}(D)} ID_P$ for all $I \in \mathbf{F}(D)$ by Lemma 0(2). We next use this result to give another interesting characterization of $*_w$-operations.

Lemma 3. *Let $*$ be a star operation on D, and let $\Lambda = *_f\text{-Max}(D) \cap \Omega$.*

(1) *$\Lambda = \emptyset$ if and only if $*_w = w$.*
(2) *If $\Lambda \neq \emptyset$, then $E^{*_w} = \bigcap_{M \in \Lambda} E^{*_M}$ for all $E \in \mathbf{F}(D)$. In particular, $*_w \lneq w$.*

Proof. (1) Assume that $\Lambda = \emptyset$. Then, since $*_f \leqslant t$, each maximal $*_f$-ideal is a t-ideal. Hence $*_f\text{-Max}(D) = t\text{-Max}(D)$, and thus $*_w = w$. The converse is clear.

(2) By Lemma 1, $ED_P \cap E^w = E^w$ for all $P \in *_f\text{-Max}(D) \setminus \Lambda$. Thus we have

$$\begin{aligned}
E^{*_w} &= \bigcap_{P \in *_f\text{-Max}(D)} ED_P \\
&= \left(\bigcap_{P \in *_f\text{-Max}(D)\setminus\Lambda} ED_P\right) \cap \left(\bigcap_{M\in\Lambda} ED_M\right) \cap E^w \\
&= \left(\bigcap_{P \in *_f\text{-Max}(D)\setminus\Lambda} \left(ED_P \cap E^w\right)\right) \cap \left(\bigcap_{M\in\Lambda} \left(ED_M \cap E^w\right)\right) \\
&= E^w \cap \left(\bigcap_{M\in\Lambda} E^{*_M}\right) \\
&= \bigcap_{M\in\Lambda} E^{*_M}.
\end{aligned}$$

The "in particular" part follows because $M^{*_w} = M \neq D = M^w$ for all $M \in \Lambda$. □

For a nonempty set Δ of nonzero prime ideals of D, let

$$E^{*_\Delta} = \bigcap_{P \in \Delta} E^{*_P}$$

for all $E \in \mathbf{F}(D)$. It is clear that $*_\Delta$ is a stable star operation on D because each $*_P$ is stable by Lemma 1. Moreover, if Δ is finite, then $*_\Delta$ is of finite type. In particular, if $\Delta = \{P\}$ is a singleton set, then $*_\Delta = *_P$. Also, by Lemma 3(1), it is reasonable to denote by $*_\Delta$ the w-operation w on D when $\Delta = \emptyset$.

Theorem 4.

(1) *If $*$ is a stable star operation of finite type, then $* = *_\Delta$ for a subset Δ of Ω.*
(2) *$|\Omega| + 1 \leqslant |SF_s(D)| \leqslant 2^{|\Omega|}$.*

Proof. (1) This follows directly from Lemmas 0(3) and 3.

(2) By (1), we have $|SF_s(D)| \leqslant 2^{|\Omega|}$. To prove that $|\Omega| + 1 \leqslant |SF_s(D)|$, we first note that if $\Omega = \emptyset$, then $t\text{-Max}(D)$ is the set of maximal ideals of D. Hence $d = w$, and thus $SF_s(D) = \{d\}$; so $|\Omega| + 1 \leqslant 1 = |SF_s(D)|$. Next, assume that $\Omega \neq \emptyset$. Then $*_M \in SF_s(D)$, $w \neq *_M$ and $*_{M_1} \neq *_{M_2}$ for all $M, M_1, M_2 \in \Omega$ with $M_1 \neq M_2$ by Lemmas 1 and 2. Thus $|\Omega| + 1 \leqslant |SF_s(D)|$. □

Let X be an indeterminate over D, $D[X]$ be the polynomial ring over D, and $\Omega(D[X])$ be the set of nonzero prime ideals Q of $D[X]$ with $Q^t = D[X]$. If D is a field, then $D[X]$ is a PID, and so $|SF_s(D[X])| = 1$. But if D is not a field, then $|\Omega(D[X])| = \infty$ (note that if P is a maximal t-ideal of D, then $P[X]$ is a maximal t-ideal (cf. [7, Proposition 1.1]) but $D[X]/P[X] \cong (D/P)[X]$ has infinitely many prime ideals). Thus $|SF_s(D[X])| = \infty$ by Theorem 4(2). In fact, by Theorem 4(2), $|\Omega| = \infty$ if and only if $|SF_s(D)| = \infty$. So when we compute $|SF_s(D)|$, we are mainly interested in integral domains D with Ω finite.

Corollary 5. *If $|\Omega| < \infty$, then*

(1) *$|SF_s(D)| = |\Omega| + 1$ if and only if Ω is linearly ordered under inclusion.*

(2) $|SF_s(D)| = 2^{|\Omega|}$ *if and only if each pair of elements in* Ω *are incomparable.*

Proof. (1) Assume to the contrary that there are $M_1, M_2 \in \Omega$ such that $M_i \nsubseteq M_j$ for $i, j = 1, 2$. Let $\Delta = \{M_1, M_2\}$. Then $(M_1 \cap M_2)^{*_\Delta} = M_1 \cap M_2 \subsetneq D = (M_1 \cap M_2)^w$, and so $*_\Delta \neq w$. Next, let $M \in \Omega$. Clearly, $*_\Delta \neq *_{M_1}, *_{M_2}$, and so we assume $M \neq M_1, M_2$. If $M_1 \cap M_2 \subseteq M$, then $M \nsubseteq M_i$ for $i = 1, 2$, and so $M^{*_\Delta} = D \neq M = M^{*_M}$; hence $*_\Delta \neq *_M$. If $M_1 \cap M_2 \nsubseteq M$, then $(M_1 \cap M_2)^{*_M} = D \neq M_1 \cap M_2 = (M_1 \cap M_2)^{*_\Delta}$; so $*_M \neq *_\Delta$. Thus $|SF_s(D)| \geqslant |\Omega| + 2$, a contradiction. The converse follows directly from Lemma 2 and Theorem 4(1).

(2) ($\Rightarrow$) Let $M_1, M_2 \in \Omega$ be such that $M_1 \subsetneq M_2$. Then $*_{M_1} \gneq *_{M_2}$ by Lemma 2, and hence $E^{*_{M_2}} = E^{*_{M_1}} \cap E^{*_{M_2}}$ for all $E \in \mathbf{F}(D)$. Thus $*_{M_2} = *_{\{M_1, M_2\}}$, which implies that $|SF_s(D)| \leqslant 2^{|\Omega|} - 1 \lneq 2^{|\Omega|}$. ($\Leftarrow$) For the converse, let Λ and Δ be two distinct subsets of Ω (for convenience, assume $\Lambda \nsubseteq \Delta$). Choose $M \in \Lambda \setminus \Delta$. Then $M^{*_\Delta} = \bigcap_{P \in \Delta} M^{*_P} = D \neq M = M^{*_\Lambda}$ because M is not comparable to each prime ideal in Δ. Thus $*_\Delta \neq *_\Lambda$. Hence $2^{|\Omega|} \leqslant |SF_s(D)|$, and therefore $2^{|\Omega|} = |SF_s(D)|$ by Theorem 4(2). □

The next corollary is an easy consequence of Theorem 4; so we omit the proof.

Corollary 6. $|SF_s(D)| = 2$ *if and only if* D *has a unique maximal ideal that is not a* t*-ideal.*

We mean by t-$\dim(D) = 1$ that D is not a field and each prime t-ideal of D is a maximal t-ideal. Examples of integral domains with t-$\dim(D) = 1$ include Krull domains and one dimensional integral domains.

Corollary 7. *(Cf.* [6, Corollary 2.11(2)]*.) If* t*-dim*$(D) = 1$*, then* $|SF_s(D)| = 2$ *if and only if* $\dim(D) = 2$ *and* D *has a unique maximal ideal of height two.*

Proof. By Corollary 6, D has a unique maximal ideal that is not a t-ideal. But, note that each height one prime ideal is a t-ideal; so the maximal ideal must be of height two. □

It is not easy in general to compute the exact value of $|SF_s(D)|$, because there are distinct nonempty subsets Δ and Λ of Ω such that $*_\Delta = *_\Lambda$ (for example, if $M_1, M_2 \in \Omega$ with $M_1 \subsetneq M_2$, then $\{M_1, M_2\} \neq \{M_2\}$ but $*_{\{M_1, M_2\}} = *_{\{M_2\}}$). We close this paper by giving an answer to the question when $*_\Delta \neq *_\Lambda$.

Proposition 8. *Let* Δ *and* Λ *be two distinct nonempty subsets of* Ω*. Then* $*_\Delta = *_\Lambda$ *if and only if* (i) *for each* $P \in \Delta$*, there is a* $Q \in \Lambda$ *such that* $P \subseteq Q$ *and* (ii) *for each* $Q' \in \Lambda$*, there is a* $P' \in \Delta$ *such that* $Q' \subseteq P'$*.*

Proof. ($\Rightarrow$) Assume to the contrary that (i) does not hold. Then there is a prime ideal $P \in \Delta$ such that $P \nsubseteq Q$ for all $Q \in \Lambda$. Hence $P^{*_\Lambda} = \bigcap_{Q \in \Lambda} P^{*_Q} = \bigcap_{Q \in \Lambda} (D_Q \cap P^w) = D \neq P = P^{*_\Delta}$. Thus $*_\Delta \neq *_\Lambda$. By the same way, we have that if (ii) does not hold, then $*_\Delta \neq *_\Lambda$. ($\Leftarrow$) Let $E \in \mathbf{F}(D)$. For $Q \in \Lambda$, let P_Q be a prime ideal in Δ such that $Q \subseteq P_Q$. Then by Lemma 2, $E^{*_\Lambda} = \bigcap_{Q \in \Lambda} E^{*_Q} \supseteq \bigcap_{Q \in \Lambda} E^{*_{P_Q}} \supseteq \bigcap_{P \in \Delta} E^{*_P} = E^{*_\Delta}$. Also, by (ii), we have $E^{*_\Lambda} \subseteq E^{*_\Delta}$. Thus $E^{*_\Delta} = E^{*_\Lambda}$. □

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