Algebra/Lie Algebras

Every monomorphism of the Lie algebra of triangular polynomial derivations is an automorphism

Tout homomorphisme injectif de l'algèbre de Lie des dérivations triangulaires polynomiales est un automorphisme

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1. Introduction

Throughout, \( K \) is a field of characteristic zero and \( K^* \) is its group of units; \( P_n := K[x_1, \ldots, x_n] = \bigoplus_{\alpha \in \mathbb{N}^n} K^{x_\alpha} \) is a polynomial algebra over \( K \) where \( x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \); \( \partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n} \) are the partial derivatives (\( K \)-linear derivations) of \( P_n \); \( \text{Aut}_K(P_n) \) is the group of automorphisms of the polynomial algebra \( P_n \); \( \text{Der}_K(P_n) = \bigoplus_{i=1}^n P_n \partial_i \) is the Lie algebra of \( K \)-derivations of \( P_n \); \( A_n := K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} K^{x^{\alpha} \partial^{\beta}} \) is the \( n \)th Weyl algebra; for each natural number \( n \geq 2 \),

\[ u_n := K \partial_1 + P_1 \partial_2 + \cdots + P_{n-1} \partial_n \]

is the Lie algebra of unitriangular polynomial derivations (it is a Lie subalgebra of the Lie algebra \( \text{Der}_K(P_n) \)) and \( G_n := \text{Aut}_K(u_n) \) is its group of automorphisms; \( \delta_1 := \text{ad}(\partial_1), \ldots, \delta_n := \text{ad}(\partial_n) \) are the inner derivations of the Lie algebra \( u_n \) determined by the elements \( \partial_1, \ldots, \partial_n \) (where \( \text{ad}(a)(b) := [a, b] \)).

The aim of the Note is to prove the following theorem:

**Theorem 1.1.** Every monomorphism of the Lie algebra \( u_n \) is an automorphism.

**Remark.** Not every epimorphism of the Lie algebra \( u_n \) is an automorphism. Moreover, there are countably many distinct ideals \( \{I_{i,n-1} \mid i \geq 0\} \) such that
1.1. The derived series for the Lie algebra \( nilpotent \) derivation. The group \( An \) algebra endomorphism of \( (see \ also \ [1] \ for \ a \ short \ proof). \) The Jacobian Conjecture claims that isomorphic to an iterated semi-direct product \( [4, \ Theorem \ 5.3], \) polynomial automorphisms, \( \) Theorem 1.2. \( \) is an automorphism \( \) and the Lie algebras \( u_n/I_{ad}^n \) and \( u_n \) are isomorphic \( [3, \ Theorem \ 5.1(1)]. \)

Theorem 1.1 has bearing of the Conjecture of Dixmier \( [6] \) for the Weyl algebra \( An \) over a field of characteristic zero that claims: every homomorphism of the Weyl algebra is an automorphism. The Weyl algebra \( An \) is a simple algebra, so every algebra endomorphism of \( An \) is a monomorphism. This conjecture is open since 1968 for all \( n \geq 1 \). It is stably equivalent to the Jacobian Conjecture for the polynomial algebras as was shown by Tsuchimoto \( [7] \), Belov-Kanel and Kontsevich \( [5] \) (see also \( [1] \) for a short proof). The Jacobian Conjecture claims that certain monomorphisms of the polynomial algebra \( P_n \) are isomorphisms: Every algebra endomorphism \( \sigma \) of the polynomial algebra \( P_n \) such that \( J(\sigma) := det(\frac{\partial^2(x_i)}{\partial x_j}) \in K^* \) is an automorphism. The condition that \( J(\sigma) \in K^* \) implies that the endomorphism \( \sigma \) is a monomorphism.

An analogue of the Conjecture of Dixmier is true for the algebra \( x_I := K[x, \frac{d}{dx}, f] \) of polynomial integro-differential operators.

Theorem 1.2. (See \( [2, \ Theorem \ 1.1]. \)) Each algebra endomorphism of \( x_I \) is an automorphism.

In contrast to the Weyl algebra \( A_1 = K(x, \frac{d}{dx}) \), the algebra of polynomial differential operators, the algebra \( x_I \) is neither a left/right Noetherian algebra nor a simple algebra. The left localizations, \( A_1, x_1, \) and \( x_I, x_1, \) of the algebras \( A_1 \) and \( x_I \) at the powers of the element \( \partial = \frac{d}{dx} \) are isomorphic. For the simple algebra \( A_1, x_1 \simeq x_1, \) there are algebra endomorphisms that are not automorphisms \( [2] \).

Before giving the proof of Theorem 1.1, let us recall several results that are used in the proof.

1.1. The derived series for the Lie algebra \( u_n \)

Let \( \mathcal{G} \) be a Lie algebra over the field \( K \) and \( a, b \) be its ideals. The commutant \( [a, b] \) of the ideals \( a \) and \( b \) is the linear span in \( \mathcal{G} \) of all the elements \( [a, b] \) where \( a \in a \) and \( b \in b \). Let \( \mathcal{G}(0) := \mathcal{G}, \mathcal{G}(1) := [\mathcal{G}, \mathcal{G}] \) and \( \mathcal{G}(i) := [\mathcal{G}(i-1), \mathcal{G}(i-1)] \) for \( i \geq 2 \). The descending series of ideals of the Lie algebra \( \mathcal{G} \),

\[
\mathcal{G}(0) = \mathcal{G} \supseteq \mathcal{G}(1) \supseteq \cdots \supseteq \mathcal{G}(i) \supseteq \mathcal{G}(i+1) \supseteq \cdots
\]

is called the derived series for the Lie algebra \( \mathcal{G} \). The Lie algebra \( u_n \) admits the finite strictly descending chain of ideals

\[
\begin{align*}
\quad u_{n,1} &:= u_0 \supseteq u_{n,2} \supseteq \cdots \supseteq u_{n,i} \supseteq \cdots \supseteq u_{n,n} \supseteq u_{n,n+1} := 0 \\
\quad [u_{n,i}, u_{n,j}] &\subseteq \begin{cases} u_{n,i+1} & \text{if } i = j, \\ u_{n,j} & \text{if } i < j. \end{cases}
\end{align*}
\]

(1)

where \( u_{n,i} := \sum_{j=1}^{n} P_{j-1} \partial_j \) for \( i = 1, \ldots, n \). For all \( i < j \).

Proposition 2.1(2) of \( [3] \) states that (1) is the derived series for the Lie algebra \( u_n \), i.e., \( (u_n)(i) = u_{n,i+1} \) for all \( i \geq 0 \).

1.2. The group of automorphisms of the Lie algebra \( u_n \)

In \( [4] \), the group of automorphisms \( G_n \) of the Lie algebra \( u_n \) of triangular polynomial derivations is found (\( n \geq 2 \)), it is isomorphic to an iterated semi-direct product \( [4, \ Theorem \ 5.3], \)

\[ T^n \ltimes \left( \text{UAut}_K(P_n)_n \times (F_n \times E_n) \right) \]

where \( T^n \) is an algebraic \( n \)-dimensional torus, \( \text{UAut}_K(P_n)_n \) is an explicit factor group of the group \( \text{UAut}_K(P_n) \) of unitriangular polynomial automorphisms, \( F_n \) and \( E_n \) are explicit groups that are isomorphic respectively to the groups \( \mathcal{I} \) and \( \mathcal{J}^{n-2} \) where \( \mathcal{I} := (1 + t^2 K[[t]]) \simeq K^N \) and \( \mathcal{J} := (K[[t]])^+ \simeq K^N. \) It is shown that the adjoint group of automorphisms \( A(u_n) \) of the Lie algebra \( u_n \) is equal to the group \( \text{UAut}_K(P_n)_n \) \( [4, \ Theorem \ 7.1]. \) Recall that the adjoint group \( \text{Ad}(\mathcal{G}) \) of a Lie algebra \( \mathcal{G} \) is generated by the elements \( \text{Ad}(g) := \sum_{i \geq 0} g^i \in \text{Aut}(\mathcal{G}) \) where \( g \) runs through all the locally nilpotent elements of the Lie algebra \( \mathcal{G} \) (an element \( g \) is a locally nilpotent element if the inner derivation \( \text{ad}(g) := [g, \cdot] \) of the Lie algebra \( \mathcal{G} \) is a locally nilpotent derivation). The group \( \text{Gn} \) contains the semi-direct product \( T^n \ltimes T_n \) where

\[ T_n := \{ \sigma \in \text{Aut}_K(P_n) \mid (x_1) = x_1, \sigma(x_i) = x_i + ai \text{ where } ai \in (x_1, \ldots, x_i-1), i = 2, \ldots, n \} \]

where \( (x_1, \ldots, x_{i-1}) \) is the maximal ideal of the polynomial algebra \( P_{i-1} := K[x_1, \ldots, x_{i-1}] \) generated by the elements \( x_1, \ldots, x_{i-1}. \)
2. Proof of Theorem 1.1

Let $\varphi : u_0 \to u_n$ be a monomorphism of the Lie algebra $u_n$. By Proposition 2.1(2) of [3], $(u_n)_i = u_{n, i+1}$ for all $i$. So, the descending chain of ideals (1) is the derived series for the Lie algebra $u_n$ of length $l(u_n) = n$ (by definition, this is the number of nonzero terms in the derived series). Clearly, $l(u_{n, 2}) = n - 1$ and

$$l(\varphi(u_i)) = l(u_n) = n$$

$(\varphi(u_n) \cong u_n)$. It follows that

$$\varphi(u_n) \nsubseteq u_{n, 2}$$

since otherwise we would have $n = l(\varphi(u_n)) \leq l(u_{n, 2}) = n - 1$, a contradiction. This means that $\partial_i' := \varphi(\partial_i) = \lambda_i \partial_i + u_i$ for some $\lambda_i \in K^*$ and $u_i \in u_{n, 2}$. We use induction on $i$ to show that

$$\partial_i' := \varphi(\partial_i) = \lambda_i \partial_i + u_i, \quad i = 1, \ldots, n, \quad (3)$$

for some elements $\lambda_i \in K^*$ and $u_i \in u_{n, i+1}$. In particular, $\partial_i = \partial_{u, \lambda_i}$. The initial step, $i = 1$, has already been established. Suppose that $i \geq 2$ and that (3) holds for all numbers $i' < i$. Since $\varphi(u_{n, i}) = u_{n, i+1}$ for all $j \geq 1$, we have the inclusion $\varphi(u_{n, i}) = \varphi(u_{n, i}) \subseteq (u_{n, i-1}) = u_{n, i}$ which implies that $\partial_i' = \lambda_i \partial_i + u_i$ for some elements $\lambda_i \in P_{i-1}$ and $u_i \in u_{n, i+1}$. It remains to show that $\lambda_i \in K^*$. This fact follows from the commutation relations $[\partial_j', \partial_i'] = 0$ for $j = 1, \ldots, i - 1$ ($0 = \varphi([\partial_j, \partial_i]) = [\partial_j', \partial_i']$). In more detail, for $j = i - 1$,

$$0 = \partial_i' = \partial_{i-1} + u_{i-1}$$

for some element $v_{i-1} \in u_{n, i+1}$. Therefore, $\partial_{i-1}(\lambda_i) = 0$, i.e., $\lambda_i \in P_{i-2}$. Now, we use a second downward induction on $j$ starting on $j = i - 1$ to show that

$$\lambda_i \in P_j \quad \forall j = 1, \ldots, i - 1. \quad (4)$$

The initial step, $j = i - 1$, has been just proved. Suppose that (4) is true for all $j = k, \ldots, i - 1$. In particular, $\lambda_i \in P_k = K[x_1, \ldots, x_k]$. We have to show that $\lambda_i \in P_{k-1}$. For, we use the equality $[\partial_k', \partial_i'] = 0$:

$$0 = [\lambda_k \partial_i + u_k, \lambda_i \partial_i + u_i = \lambda_k \partial_k \partial_i + v_k$$

for some element $v_k \in u_{n, i+1}$. Since $\lambda_i \in P_k$ and $\varphi(u_{n, i}) = u_{n, i+1}$, we have $\varphi(u_{n, i}) = u_{n, i+1}$ and $\varphi(u_{n, i}) \subseteq u_{n, i}$. The monomorphism $\varphi$ respects the Lie subalgebra $G = K \partial_i + u_{n, i+1}$ of the Lie algebra $u_n$, i.e., $\varphi(G) \subseteq G$. The inclusion of Lie algebras $G \subseteq u_{n, i}$ yields the inequality $l(G) \leq l(u_{n, i}) = n - i + 1$ (the equality follows from the fact that $(u_{n, i})_{i+1} = u_{n, i+1}$ for all $i \geq 0$). The vector space

$$H = K \partial_i + K[x_1] \partial_{i+1} + K[x_1, x_{i+1}] \partial_{i+2} + \cdots + K[x_1, \ldots, x_{n-1}] \partial_n$$

is a Lie subalgebra of $G$ which is isomorphic to the Lie algebra $u_{n, i+1}$, therefore, $l(H) = l(u_{n, i+1}) = n - i + 1$. The inclusion of Lie algebra $H \subseteq G$ yields the inequality $n - i + 1 = l(H) \leq l(G)$. Therefore, $l(G) = n - i + 1$.

Suppose that $\lambda_i = 0$, we seek a contradiction. In that case, $\varphi(G) \subseteq u_{n, i+1}$ and so

$$n - i + 1 = l(G) = l(\varphi(G)) \leq l(u_{n, i+1}) = n - i$$

a contradiction.

Therefore, (3) holds. By Theorem 3.6(2) of [4], there exists a unique automorphism $\sigma \in \mathbb{T}^n \times \mathbb{T}^n \cong G_n$ such that $\sigma(\partial_i) = \partial_i'$ for $i = 1, \ldots, n$. By replacing the monomorphism $\varphi$ by the monomorphism $\varphi^{-1} \sigma$, without loss of generality we can assume that

$$\partial_i' = \partial_i \quad \forall i = 1, \ldots, n.$$
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References