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Geodesics in infinite dimensional Stiefel and Grassmann manifolds ☆

Géodesiques sur des variétiés de Stiefel et de Grassmann de dimension infinie

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ABSTRACT

Let V be a separable Hilbert space, possibly infinite dimensional. Let $\mathbf{St}(p, V)$ be the Stiefel manifold of orthonormal frames of p vectors in V, and let $\mathbf{Gr}(p, V)$ be the Grassmann manifold of p-dimensional subspaces of V. We study the distance and the geodesics in these manifolds, by reducing the matter to the finite dimensional case. We then prove that any two points in those manifolds can be connected by a minimal geodesic, and characterize the cut locus.

RÉSUMÉ

Soit V un espace de Hilbert séparable, éventuellement de dimension infinie. Soient $\mathbf{St}(p, V)$ l'ensemble des systèmes orthonormés de p vecteurs de V, appelé la variété de Stiefel, et $\mathbf{Gr}(p, V)$ l'ensemble des sous-espaces vectoriels de V de dimension p, appelé la variété Grassmannienne. En réduisant le problème en dimension finie, nous montrons que dans ces espaces il existe des géodésiques minimales entre chaque paire de points et nous caractérisons le cut-locus.

1. Introduction

Let V be a separable Hilbert space, let p be a positive natural number. We assume that $\dim(V) \geqslant (2p)$ from here on. $\mathbf{St}(p, V)$ is the set of orthonormal frames of p vectors in V. Equivalently, we consider

$$\mathbf{St}(p, V) = \left\{x \in L\left(\mathbb{R}^p, V\right):\ x^\top \circ x = \mathrm{Id}_{\mathbb{R}^p}\right\}$$

to be the set of all linear isometric immersions of $\mathbb{R}^p$ into V. Here $x^\top \in L(V, \mathbb{R}^p)$ is the transpose with respect to the metrics on V and $\mathbb{R}^p$, i.e.

$$\left\langle x^\top(v), r\right\rangle_{\mathbb{R}^p} = \left\langle v, x(r)\right\rangle_V \quad \text{for all } v \in V,\ r \in \mathbb{R}^p.$$

$\mathbf{St}(p, V)$ is a smooth embedded submanifold in V^p. The induced Riemannian metric on $\mathbf{St}(p, V)$ is $\langle x, y\rangle = \mathrm{tr}(x^\top y)$. $\mathbf{St}(p, V)$ is a complete Riemannian manifold with this metric. $\mathbf{Gr}(p, V)$ is the manifold of p-dimensional linear subspaces of V and equals the orbit space $\mathbf{St}(p, V)/O(p)$ with respect to $O(p)$ acting on $\mathbf{St}(p, V)$ by composition from the right.

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Our interest is due to the fact that $\mathbf{St}(2, V)$ with $V = L^2([0, 1])$ is isometric to the space of planar closed curves up to translation and scaling, endowed with a Sobolev metric of order one. The $O(2)$-action on $\mathbf{St}(2, V)$ corresponds to rotations of the curves. Thus $\mathbf{Gr}(2, V)$ with $V = L^2([0, 1])$ is isometric to the space of planar closed curves up to translations, scalings and rotations. See [7,8,5,6]. Any result that is proven about the Stiefel or Grassmannian immediately carries over to the corresponding space of curves.

2. Critical geodesics

We will call a curve γ in a Riemannian manifold a **critical geodesic** if it is a solution to the equation $\nabla_{\partial_t}\dot\gamma = 0$, where ∇ is the covariant derivative. Such γ is a critical point for the action $\int_0^1 g(\dot\gamma, \dot\gamma)\,\mathrm{d}t$.

Proposition 1 *(Critical geodesics in $\mathbf{St}(p, V)$). Let $\mathbf{St}(p, V)$ be endowed with the induced metric from V^p. Let $\gamma : [0, 1] \to \mathbf{St}(p, V)$ be a path. Then the geodesic equation is $\ddot\gamma + \gamma(\dot\gamma^\top\dot\gamma) = 0$. Solutions to the geodesic equation exist for all time and are given by*

$$\big(\gamma(t)e^{At}, \dot\gamma(t)e^{At}\big) = \big(\gamma(0), \dot\gamma(0)\big)\exp t\begin{pmatrix} A & -S \\ \mathrm{Id} & A \end{pmatrix} \tag{1}$$

where $A = \gamma(0)^\top\dot\gamma(0)$, $S = \dot\gamma(0)^\top\dot\gamma(0)$, and Id *is the $p \times p$ identity matrix.*

For $V = \mathbb{R}^n$ this has been demonstrated by Edelman et al. [2, Section 2.2.2].[1] Going through their proof one sees that the same result holds when V is infinite dimensional.

Proposition 2. *Eq. (1) shows that the subspace of V spanned by the $(2p)$ columns of $\gamma(t)$, $\dot\gamma(t)$ remains in the space spanned by the columns of $\gamma(0)$, $\dot\gamma(0)$ for all t.*

This means that, if W is the subspace of V spanned by the columns of $\gamma(0)$, $\dot\gamma(0)$, then we can formulate the geodesic equation as an equation in $\mathbf{St}(2, W)$. Obviously, $\dim(W) \leqslant 2p$.

This also means that, if γ is a critical geodesic connecting x to y, and the space W spanned by the columns of x, y is $(2p)$-dimensional, then, for any t, the columns of $\gamma(t)$ and of $\dot\gamma(t)$ must be contained in W.

3. Minimal geodesics

We denote by $d(x, y)$ the infimum of the length of all paths connecting two points x, y in a Riemannian manifold. It does not matter whether the infimum is taken over smooth or absolutely continuous paths.[2] We call a path γ a **minimal geodesic** if its length is equal to the distance $d(\gamma(0), \gamma(1))$. Up to a time reparametrization, a minimal geodesic is smooth and is a critical geodesic. We will always silently assume that minimal geodesics are parametrized such that they are critical.

Let (M, g) be a Riemannian manifold, and d be the induced distance. When M is finite dimensional, by the celebrated Hopf–Rinow theorem, metric completeness of (M, d) is equivalent to geodesic completeness of (M, g), and both imply that any two points $x, y \in M$ can be connected by a minimal geodesic. In infinite dimensional manifolds this is not true in general. Indeed, in [1] there is an example of an infinite dimensional metrically complete Hilbert smooth manifold M and $x, y \in M$ such that there is no critical (and thus no minimal) geodesic connecting x to y. A simpler example, due to Grossman [3] (see also Section VIII §6 in [4]), is an infinite dimensional ellipsoid where the south and north pole can be connected by countably many critical geodesics of decreasing length, so that the distance between the poles is not attained by any minimal geodesic.

We will show that, even when V is infinite dimensional, any two points in $\mathbf{St}(p, V)$ and $\mathbf{Gr}(p, V)$ can be connected by a minimal geodesic.

3.1. Minimal geodesics in the Stiefel manifold

Theorem 3. *Let V be a Hilbert space. Consider a $(2p)$-dimensional Hilbert space W and an isometric linear embedding $i : W \to V$. Then i induces an isometric embedding*

$$i_* : \mathbf{St}(p, W) \to \mathbf{St}(p, V), \qquad x \mapsto i \circ x$$

(here we consider $x \in \mathbf{St}(p, W)$ to be a linear isometric immersion of $\mathbb{R}^p$ into W).

(1) *$i_*(\mathbf{St}(p, W))$ is totally geodesic in $\mathbf{St}(p, V)$.*

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[1] [2] credits a personal communication by R.A. Lippert for the final closed form formula (1).

[2] Lemma 6.1 in Chap. VIII in [4] can be used to convert any absolutely continuous path to a shorter and piecewise smooth path.

(2) *Let d_W be the distance in* $\mathbf{St}(p, W)$ *and similarly d_V in* $\mathbf{St}(p, V)$*, then*

$$d_W(x, y) = d_V\big(i_*(x), i_*(y)\big). \tag{2}$$

(3) *Let $x, y \in \mathbf{St}(p, W)$, and a minimal geodesic γ connecting x to y in $\mathbf{St}(p, W)$: then $i_* \circ \gamma$ is a minimal geodesic connecting $i_*(x)$ to $i_*(y)$ in $\mathbf{St}(p, V)$.*
(4) *The diameter of $\mathbf{St}(p, V)$ is equal to the diameter of $\mathbf{St}(p, \mathbb{R}^{2p})$.*
(5) *Any two points $x, y \in \mathbf{St}(p, V)$ can be connected by a minimal geodesic γ. Any minimal geodesic lies in $\mathbf{St}(p, U)$, where U is a $(2p)$-dimensional subspace of V (dependent on γ).*
(6) *Let $x, y \in \mathbf{St}(p, V)$. Then y is in the cut locus of x if and only if there is a $(2p)$-dimensional subspace W of V and $\tilde{x}, \tilde{y} \in \mathbf{St}(p, W)$ such that $x = i_*(\tilde{x})$, $y = i_*(\tilde{y})$ and $i_*(y)$ is in the cut locus of $i_*(x)$.*

Note that point (5) in the above theorem implies that minimal geodesics can be numerically computed using a finite dimensional algorithm; see Section 3.3.4 in [6].

We will need two lemmas.

Lemma 4. *Given $x \in \mathbf{St}(p, V)$, the set of $y \in \mathbf{St}(p, V)$ such that the columns of x, y are linearly independent is dense in $\mathbf{St}(p, V)$.*

Proof. Let U be the linear space spanned by the columns of x, y; if this space is not $(2p)$-dimensional, then let $r_1, \ldots, r_k$ be orthonormal vectors that lie in $U^\perp$, with $k = 2p - \dim(U)$. Up to reindexing the columns of y, we can suppose that the columns $x_1, \ldots, x_p$, $y_1, \ldots, y_{p-k}$ are linearly independent. For $\varepsilon > 0$ small, we then define

$$\tilde{y}_i = \begin{cases} y_i & i = 1, \ldots, p-k \\ \cos(\varepsilon) y_i + \sin(\varepsilon) r_i & i = (p-k+1), \ldots, p. \end{cases}$$

It is easy to verify that $\tilde{y} \in \mathbf{St}(p, V)$ and that the columns of $x, \tilde{y}$ are linearly independent. □

Lemma 5. *The theorem holds when V is a Hilbert space of finite dimension n with $n > 2p$.*

Proof. We prove point (1). Let us consider the subgroup $G = O(i(W)^\perp)$ of $O(V)$ that keeps $i(W)$ fixed. Then G acts isometrically on $\mathbf{St}(p, V)$ as well, and its fixed point set is $i_*(\mathbf{St}(p, W))$. This proves that $i_*(\mathbf{St}(p, W))$ is totally geodesic in $\mathbf{St}(p, V)$.

To prove point (2) we first note that since $\mathbf{St}(p, W)$ is isometrically embedded in $\mathbf{St}(p, V)$, we have $d_W(x, y) \geqslant d_V(i_*(x), i_*(y))$. We will show the inverse inequality only for the case when the columns of x and y are linearly independent. The general case then follows because the set of y such that the columns of x and y are linearly independent is dense in W by Lemma 4 and since distances are Lipschitz continuous.

Since V is finite dimensional, $\mathbf{St}(p, V)$ is compact, so by the Hopf–Rinow theorem $i_*(x)$ and $i_*(y)$ can be connected by a minimizing geodesic in $\mathbf{St}(p, V)$. The columns of $i_*(x)$ and $i_*(y)$ together span the $(2p)$-dimensional space $i(W)$, so we can apply Proposition 2. This allows us to write $\gamma = i_* \circ \tilde{\gamma}$ for a path $\tilde{\gamma}$ in $\mathbf{St}(p, W)$. Then

$$d_W(x, y) \leqslant \operatorname{len}(\tilde{\gamma}) = \operatorname{len}(i_* \circ \tilde{\gamma}) = \operatorname{len}(\gamma) = d_V\big(i_*(x), i_*(y)\big).$$

Point (3) follows from point (2) and the equality

$$\operatorname{len}(i_* \circ \gamma) = \operatorname{len}(\gamma) = d_W(x, y) = d_V\big(i_*(x), i_* y\big).$$

Point (4) follows from point (3). Point (5) follows from the Hopf–Rinow theorem and the discussion in Proposition 2.

We now prove point (6). By definition, y is in the cut locus of x if and only if there is a geodesic γ in $\mathbf{St}(p, V)$ with $\gamma(0) = x$, $\gamma(1) = y$ such that

$$\sup\big\{t\colon \operatorname{len}(\gamma|_{[0,t]}) = d_V\big(\gamma(0), \gamma(t)\big)\big\} = 1.$$

(Recall that we write d_V for the distance in $\mathbf{St}(p, V)$.) Any such geodesic lies in $\mathbf{St}(p, W)$ for some $(2p)$-dimensional space W. Letting $i : W \to V$ denote the isometric embedding, we can write $\gamma = i_* \circ \tilde{\gamma}$ for a path $\tilde{\gamma}$ in $\mathbf{St}(p, W)$. Then one has by point (2) that

$$\sup\big\{t\colon \operatorname{len}(\tilde{\gamma}|_{[0,t]}) = d_W\big(\tilde{\gamma}(0), \tilde{\gamma}(t)\big)\big\} = 1. \qquad \square$$

We now prove Theorem 3.

Proof. The proof of points (1), (3), (4), (6) works as in the finite dimensional case. We will now prove point (2). We have $d_W(x, y) \geqslant d_V(i_*(x), i_*(y))$, since $\mathbf{St}(p, W)$ is isometrically embedded in $\mathbf{St}(p, V)$. It remains to show the inverse inequality.

Consider a smooth path ξ connecting $i_*(x)$ to $i_*(y)$ in $\mathbf{St}(p,V)$. We can find finitely many points $0=t_0<t_1<\cdots<t_k=1$ such that $\xi|_{[t_i,t_{i+1}]}$ is contained inside (the manifold part of) a normal neighborhood. So $\xi(t_i), \xi(t_{i+1})$ can be connected by a minimal geodesic. By joining all these minimal geodesics we obtain a piecewise smooth path η, with $\operatorname{len}(\eta)\leqslant\operatorname{len}(\xi)$. Then by repeated application of Proposition 2 there is a finite dimensional subspace $\tilde{W}$ of V that contains the columns of $\eta(t)$ for $t\in[0,1]$. When necessary we enlarge $\tilde{W}$ such that it also contains $i(W)$. Now the finite dimensional version of this lemma allows us to compare $\mathbf{St}(p,W)$ to $\mathbf{St}(p,\tilde{W})$, and we get:

$$d_W(x,y)=d_{\tilde{W}}\big(i_*(x),i_*(y)\big)\leqslant\operatorname{len}(\eta)\leqslant\operatorname{len}(\xi).$$

Since this holds for arbitrary paths ξ connecting $i_*(x)$ to $i_*(y)$ in $\mathbf{St}(p,V)$, we get $d_W(x,y)\leqslant d_V(i_*(x),i_*(y))$.

Point (5) now follows by choosing any linear subspace W containing the columns of x, y. □

3.2. Minimal geodesics in the Grassmann manifold

Theorem 6. *Theorem 3 remains valid when Stiefels are replaced by Grassmannians. Most importantly, for any two points $x,y\in\mathbf{Gr}(p,V)$, there is a minimal geodesic γ connecting x to y. The same holds for the Grassmannian $\mathbf{Gr}_+(p,V)$ of oriented p spaces.*

We need a lemma.

Lemma 7 *(Existence of horizontal paths). For any path $x:[0,1]\to\mathbf{St}(p,V)$ there is a path $g:[0,1]\to O(p)$ such that the path $x(t)\circ g(t)$ is horizontal, i.e. normal to the $O(p)$-orbits in $\mathbf{St}(p,V)$.*

Proof. The tangent space at x to the $O(p)$-orbit through $x\in\mathbf{St}(p,V)$ is

$$T_x\big(x.O(p)\big)=\big\{xz:\ z\in\mathfrak{o}(p)\big\}=\big\{xz:\ z\in L\big(\mathbb{R}^p,\mathbb{R}^p\big),\ z^\top+z=0\big\}.$$

Thus $y\in T_x\mathbf{St}(p,V)$ is horizontal if and only if $\operatorname{tr}(y^\top xz)=0$ for all antisymmetric z. This is equivalent to $y^\top x=0$ because $y^\top x$ is antisymmetric, too. Thus the path xg is horizontal iff

$$\big(\partial_t(xg)\big)^\top(xg)=g^\top\dot{x}^\top xg+\dot{g}^\top x^\top xg=0.$$

This can be achieved by letting g be the solution to the ODE $\dot{g}=-x^\top\dot{x}g$. □

Note that the length of $x(t)\circ g(t)$ is smaller than or equal to the length of $x(t)$, with equality if and only if $x(t)$ is already a horizontal path.

We are now able to prove Theorem 6.

Proof. $\mathbf{St}(p,V)$ is a principal fiber bundle with structure group $O(p)$ over $\mathbf{Gr}(p,V)=\mathbf{St}(p,V)/O(p)$. We prove the existence of minimizing geodesics connecting any two points in $\mathbf{Gr}(p,V)$. Take any point $\tilde{x}\in\mathbf{St}(p,V)$ in the fiber over x. The fiber over y is compact since $O(p)$ is compact. Therefore $d(x,\cdot)$ attains a minimum at some point $\tilde{y}$ in the fiber over y. By Theorem 3 there is a minimal geodesic connecting $\tilde{x}$ to $\tilde{y}$. This geodesic is horizontal since otherwise it could be made shorter by making it horizontal. (We use Lemma 7 here.) By the theory of Riemannian submersions it projects to a minimal geodesic in $\mathbf{Gr}(p,V)$.

The remaining statements simply follow from Theorem 3 by going to the quotient with respect to the $O(p)$-action. For the case of $\mathbf{Gr}_+(p,V)$, we use the group $SO(p)$ instead of $O(p)$. □

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