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Algebraic Geometry

Hodge structures and Weierstrass σ-function

Structures de Hodge et fonction σ de Weierstrass

Grzegorz Banaszak [a,1], Jan Milewski [b,2]

[a] *Department of Mathematics and Computer Science, Adam Mickiewicz University, 61-614 Poznań, Poland*
[b] *Institute of Mathematics, Poznań University of Technology, ul. Piotrowo 3A, 60-965 Poznań, Poland*

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ABSTRACT

In this Note we introduce new definition of Hodge structures and show that $\mathbb{R}$-Hodge structures are determined by $\mathbb{R}$-linear operators that are annihilated by the Weierstrass σ-function.

RÉSUMÉ

Dans cette Note, nous introduisons une nouvelle définition des structures de Hodge et démontrons que les structures de Hodge sur $\mathbb{R}$ sont déterminées par des transformations $\mathbb{R}$-linéaires qui sont des zéros de la fonction σ de Weierstrass.

1. Introduction

Classically a real Hodge structure of a given weight can be defined in four equivalent ways as follows (see e.g. [2,5]):

Definition 1.1. A real Hodge structure of a weight n consists of a finite-dimensional $\mathbb{R}$-vector space $V = V_{\mathbb{R}}$ together with any of the following equivalent data:

(i) A decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$, called the *Hodge decomposition*, such that $\overline{V^{p,q}} = V^{q,p}$, where $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$.
(ii) A decreasing filtration $F_H^r V_{\mathbb{C}}$ of $V_{\mathbb{C}}$, called the *Hodge filtration*, such that $F_H^r V_{\mathbb{C}} \oplus \overline{F_H^{n-r+1} V_{\mathbb{C}}} = V_{\mathbb{C}}$.
(iii) A homomorphism $h_n : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ of real algebraic groups, and also specifying that the weight of the Hodge structure is n, where $\mathbb{S} := R_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$.
(iv) A homomorphism $h_n : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ of real algebraic groups such that via the composition $\mathbb{G}_m/\mathbb{R} \to \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ an element $t \in \mathbb{G}_m/\mathbb{R}$ acts as $t^{-n} \cdot Id$.

Throughout the paper we work with Hodge structures of various weights, hence by a Hodge structure we understand here a finite direct sum

E-mail addresses: banaszak@amu.edu.pl (G. Banaszak), jsmilew@wp.pl (J. Milewski).

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$$\rho := \bigoplus_{j=1}^{k} h_{n_j} \tag{1}$$

of representations h_{n_j} described in (iii) or (iv) of Definition 1.1.

In this paper we consider Hodge structures on real vector space V via representations of the Lie algebra of the real algebraic group $\mathbb{S}$ (denoted also $\mathbb{C}^\times$) on V. In Section 2 we show that a Hodge structure can be treated as a pair of operators E, T on V satisfying certain conditions (see Theorem 2.1). In Section 3 we show that a Hodge structure can be treated as a single operator $S := E + T$ on V such that $\sigma(S) = 0$ for a Weierstrass σ-function which corresponds to decomposition of V into eigenspaces of the operators E and T. Weierstrass σ-function does not have multiple zeros and this corresponds to the fact that the complexification of S does not have generalized eigenvectors other than ordinary ones.

2. Hodge structures and Lie algebras

The following theorem gives another definition of the Hodge structure.

Theorem 2.1. *Let V be a finite-dimensional vector space over $\mathbb{R}$. There is a one-to-one correspondence between the family of Hodge structures on V and the family of pairs of endomorphisms $E, T \in \mathrm{End}_{\mathbb{R}}(V)$ satisfying the following conditions*:

$$[E, T] = 0, \qquad \sin(\pi E) = 0, \qquad \sinh(\pi T) = 0, \tag{2}$$

$$\sin\left(\frac{\pi}{2}\left(E^2 + T^2\right)\right) = 0. \tag{3}$$

Proof. Consider a Hodge structure on V. By (1) (cf. Definition 1.1 (iii)) this gives a representation:

$$\rho : \mathbb{S} \to \mathrm{GL}(V)$$

of real algebraic groups. The representation ρ decomposes into irreducible representations $\rho_{p,q}$ with multiplicities $m_{p,q}$

$$\rho = \bigoplus_{q \leqslant p} m_{p,q}\rho_{p,q},$$

$$\rho_{p,q}\left(re^{i\phi}\right) := r^{p+q}\begin{bmatrix} \cos(p-q)\phi & -\sin(p-q)\phi \\ \sin(p-q)\phi & \cos(p-q)\phi \end{bmatrix} \quad \text{for } p \neq q,\ \ p, q \in \mathbb{Z},$$

$$\rho_{p,p}\left(re^{i\phi}\right) := r^{2p}\,[\,1\,]. \tag{4}$$

Certainly, the complexification of the representation $\rho_{p,q}$ for $q < p$ decomposes into two one-dimensional $\mathbb{C}$-vector spaces:

$$\rho_{p,q} \otimes_{\mathbb{R}} \mathbb{C} = \rho^{\mathbb{C}}_{p,q} \oplus \rho^{\mathbb{C}}_{q,p}, \tag{5}$$

where

$$\rho^{\mathbb{C}}_{m,n}(z) = z^m \bar{z}^n. \tag{6}$$

Consider the real Lie algebra representation (the derivative of ρ):

$$\mathcal{L}(\rho) : \mathbb{C} \to \mathrm{End}_{\mathbb{R}}(V).$$

For $q < p$ the representation $\mathcal{L}(\rho_{p,q})$ is also two-dimensional

$$\mathcal{L}(\rho_{p,q})(1) = (p+q)I \quad \text{and} \quad \mathcal{L}(\rho_{p,q})(i) = (p-q)J, \tag{7}$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

For $p = q$

$$\mathcal{L}(\rho_{p,p})(1) = 2p \quad \text{and} \quad \mathcal{L}(\rho_{p,p})(i) = 0. \tag{8}$$

If we put

$$E := \mathcal{L}(\rho)(1) \quad \text{and} \quad T := \mathcal{L}(\rho)(i) \tag{9}$$

then we get Eqs. (2) and (3). The condition (3) is fulfilled because $p - q$ and $p + q$ have the same parity.

Now let us assume that conditions (2) and (3) hold. Observe that $\sinh(z)$ and $\sin(z)$ have single zeros in the complex plane. Moreover (2) and (3) imply that the complexifications $E \otimes 1$ and $T \otimes 1 \in \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ have common eigenbasis. From this it follows that the endomorphisms $E, T \in \mathrm{End}_{\mathbb{R}}(V)$ have common Jordan decomposition into eigenspaces of dimension 1 or 2. We define a representation

$$\rho : \mathbb{C}^{\times} \to \mathrm{GL}(V),$$
$$\rho\bigl(e^{x+iy}\bigr) = \exp(xE + yT) \quad \text{for } x, y \in \mathbb{R}.$$

ρ is an algebraic representation, because the equality (3) holds. The representation ρ gives the Hodge structure on V. $\square$

3. Hodge structures via single operator

Let $\sigma(z)$ be the Weierstrass sigma function for the lattice generated by $\omega_1 = 1 - i$ and $\omega_2 = 1 + i$:

$$\sigma(z) := z \prod_{(k_1,k_2)\neq(0,0)} \left(1 - \frac{z}{k_1\omega_1 + k_2\omega_2}\right) \exp\left[\frac{z}{k_1\omega_1 + k_2\omega_2} + \frac{1}{2}\left(\frac{z}{k_1\omega_1 + k_2\omega_2}\right)^2\right].$$

Theorem 3.1. *For operators $E, T \in \mathrm{End}_{\mathbb{R}}(V)$ defined in (9) let $S := E + T$. Then we get the following equality*

$$\sigma(S) = 0. \tag{10}$$

Conversely every $S \in \mathrm{End}_{\mathbb{R}}(V)$ satisfying condition (10) gives a unique pair (E, T) of operators in $\mathrm{End}_{\mathbb{R}}(V)$ such that $S = E + T$ and the conditions (2) and (3) hold.

Proof. Let

$$S_{p,q} := \mathcal{L}(\rho_{p,q})(1) + \mathcal{L}(\rho_{p,q})(i) = \begin{cases} (p+q)I + (p-q)J & \text{if } p < q, \\ 2p[\,1\,] & \text{if } p = q. \end{cases}$$

From (4), (7) and (8) we get, that $S = E + T$ has the following Jordan decomposition

$$S = \bigoplus_{q \leqslant p} m_{p,q} S_{p,q}. \tag{11}$$

Observe that $f_{p,p}(z) := z - 2p$ is the characteristic polynomial of $S_{p,p}$ and it divides the $\sigma(z)$ in the domain of analytic functions. Moreover the characteristic polynomial

$$f_{p,q}(z) := \bigl(z - (p+q) - (p-q)i\bigr)\bigl(z - (p+q) + (p-q)i\bigr)$$

of the operator $S_{p,q}$ for $q < p$, is also a factor of $\sigma(z)$ in the domain of analytic functions. So the minimal polynomial $f(z)$ of S is also a factor of the Weierstrass σ-function as a product of the form $\prod_{p,q} f_{p,q}(z)^{n_{p,q}}$, where $n_{p,q} \in \{0, 1\}$ and $n_{p,q} = 0$ for almost all (p, q). Hence $S = E + T$ satisfies Eq. (10).

Conversely, assume that an operator $S \in \mathrm{End}_{\mathbb{R}}(V)$ satisfies (10). Since the σ-function has zeros of order 1, we observe that the complexification of S is diagonalizable. We get the operators E and T considering equation

$$S(v) = \lambda v \tag{12}$$

in the complexification of V. The eigenvalues have integer real and imaginary parts with the same parity:

$$\lambda = a + ib, \quad a, b \in \mathbf{Z},\ a - b \in 2\mathbf{Z}. \tag{13}$$

Moreover we define the operators E, T in such a way that their complexifications acting on the eigenvector v of S have form: $E(v) = av$ and $T(v) = ibv$ where $S(v) = (a + ib)v$. Operators E and T satisfy Eqs. (2) and (3). The operators E and T are uniquely determined. Indeed, if $S = E' + T'$ such that E' and T' satisfy (2) and (3) then it is clear that $[E', S] = 0$ and $[T', S] = 0$. $\square$

Remark 3.2. For certain Hodge structures the set of eigenvalues of the complexification of S has further obstructions beyond (13). In this case S satisfies the equation $g(S) = 0$, where $g(z)$ is an analytic function that divides $\sigma(z)$ in such a way that $\frac{\sigma(z)}{g(z)}$ is also an analytic function on the whole complex plane.

Remark 3.3. In our work in progress we define certain deformations of Hodge structures that arise in a natural way in mathematical physics (see [1,3,4]).

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