Mathematical Analysis/Mathematical Problems in Mechanics

Asymptotically exact Korn’s constant for thin cylindrical domains

Développement asymptotique précis de la constante de Korn dans une poutre mince

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1. Introduction and results

Given a domain \( \Omega \subset \mathbb{R}^3 \), Korn’s inequality [13]:

\[
\int_{\Omega} |\nabla u|^2 \, d^3 x \leq C_K \int_{\Omega} |\mathbf{E}(u)|^2 \, d^3 x, \quad \forall u \in \mathcal{A} \subset H^1(\Omega; \mathbb{R}^3)
\]

is the key estimate to establish the solvability of the boundary-value problem of linear elastostatics [2]. This estimate holds under fairly general assumptions on \( \Omega \), provided that certain side conditions are imposed on the displacement \( u \) through the choice of the admissible space \( \mathcal{A} \) (two examples are given in (2) below). It asserts that the \( L^2 \) norm of the strain \( \mathbf{E}(u) := \text{sym} \nabla u \) controls the \( L^2 \) norm of the whole displacement gradient. The optimal choice for Korn’s constant \( C_K \) is given by \( 1/K(\Omega, \mathcal{A}) \), where

\[
K(\Omega, \mathcal{A}) := \inf_{u \in \mathcal{A} \setminus \{0\}} \frac{\int_{\Omega} |\mathbf{E}(u)|^2 \, d^3 x}{\int_{\Omega} |\nabla u|^2 \, d^3 x}.
\]

A vast body of literature investigates the dependence of Korn’s constant on the geometric properties of the domain. Estimates for thin domains, such as rods and plates, were obtained in [12,15,1,3,18,5,16,11]. Let us consider a family of rod-like domains:
$\Omega^\varepsilon = \varepsilon \omega \times (0, \ell) := \{ x' = (\varepsilon x_1, \varepsilon x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \omega, x_3 \in (0, \ell) \}$ with $\varepsilon > 0$,
and let us set
\[
\kappa_{\varepsilon}^\omega (\omega, \ell) := \frac{1}{\varepsilon^2} K (\Omega^\varepsilon, A_{\varepsilon}^\omega) = \sup_{u \in A_{\varepsilon}^\omega \setminus \{0\}} \frac{\int_{\Omega^\varepsilon} |E(u)|^2 d^3x}{\int_{\Omega^\varepsilon} |\nabla u|^2 d^3x},
\]
where the subscript "\(\varepsilon\)" stands for either "dd" or "dn", with
\[
A_{dd}^\varepsilon = \{ u \in H^1 (\Omega^\varepsilon ; \mathbb{R}^3) : u|_{x_3 = 0} = u|_{x_3 = \ell} = 0 \}, \quad A_{dn}^\varepsilon = \{ u \in H^1 (\Omega^\varepsilon ; \mathbb{R}^3) : u|_{x_3 = 0} = 0 \}.
\]
In this note we show that
\[
\lim_{\varepsilon \to 0} \kappa_{\varepsilon}^\omega (\omega, \ell) = \kappa^\omega (\omega, \ell), \quad \text{where } \kappa_{dd} (\omega, \ell) = \frac{\pi^2}{4 \ell^2} J (\omega) \quad \text{and } \kappa_{dn} (\omega, \ell) = \frac{\pi^2}{8 \ell^2} J (\omega),
\]
with
\[
J_1 (\omega) := \min_{\varepsilon \in H^1 (\omega)} \int_{\omega} (D_1 \psi - x_2)^2 + (D_2 \psi + x_1)^2 \, dx_1 \, dx_2, \quad J (\omega) := \min \left\{ J_1 (\omega), J_2 (\omega), \frac{J_1 (\omega)}{2} \right\},
\]
\[
J_1 (\omega) := \int_{\omega} x_1^2 \, dx_1 \, dx_2, \quad J_2 (\omega) := \int_{\omega} x_2^2 \, dx_1 \, dx_2, \quad A (\omega) := \int_{\omega} 1 \, dx_1 \, dx_2.
\]
We point out that, while the limit $\kappa_{dd}$ depends on the cross-section simply through the ratio $J_1 / A$, the dependence of $\kappa_{dn}$ on $\omega$ is more involved. For example, $J_1 / 2 = J_1$ for a circle, $J < J_1 / 2$ for an ellipsis, and $J_1 / 2 < J$ for a square. A detailed discussion of these examples can be found in [20].

2. Rescaling and $\Gamma$-convergence of Rayleigh’s quotient

Our proof of (3) is based on $\Gamma$-convergence. Following the standard approach [4], we perform a change of variables. To this end, we set $\Omega = \Omega^3$, and $A^\varepsilon = A_{\varepsilon}^1$. Then, to every $u \in A^\varepsilon$ we associate $v \in A^\varepsilon$ defined by $v_\alpha (x) = \varepsilon u_\alpha (x')$ and $v_3 (x) = u_3 (x')$, where $x = (x_1, x_2, x_3) \in \Omega$ and $x' = (\varepsilon x_1, \varepsilon x_2, x_3) \in \Omega^\varepsilon$. As a result, we can rewrite (1) as
\[
\kappa_{\varepsilon}^\omega (\omega, \ell) = \inf_{v \in A_{\varepsilon}^\omega \setminus \{0\}} \mathcal{R}^\omega (v), \quad \text{where } \mathcal{R}^\omega (v) := \frac{\int_{\Omega} |E^\varepsilon (v)|^2 d^3x}{\int_{\Omega} |\nabla^\varepsilon v|^2 d^3x},
\]
with
\[
[\nabla^\varepsilon v]_{\alpha \beta} = \frac{\varepsilon v_{\alpha, \beta}}{\varepsilon^2}, \quad (\nabla^\varepsilon v)_{\alpha 3} = \frac{v_{\alpha, 3}}{\varepsilon}, \quad (\nabla^\varepsilon v)_{3 \alpha} = - \frac{v_{3, \alpha}}{\varepsilon}, \quad (\nabla^\varepsilon v)_{33} = \varepsilon v_{3, 3}, \quad E^\varepsilon (v) = \text{sym } \nabla^\varepsilon v,
\]
where Greek indices run over $\{1, 2\}$, and a comma denotes partial differentiation. We next introduce the spaces:
\[
A_{BN} := \{ v \in H^1 (\Omega; \mathbb{R}^3) : E_{33} (v) = 0 \},
\]
\[
H^1_{dn} (0, \ell) := \{ f \in H^1 (0, \ell) : f (0) = 0 \}, \quad \text{and } H^1_{dd} (0, \ell) := \{ f \in H^1_{dn} (0, \ell) : f (\ell) = 0 \},
\]
and we prove:

Theorem 2.1. Let the functional $\mathcal{R} : A^\varepsilon \times H^1_{2} (0, \ell) \to \mathbb{R} \cup [+\infty]$ be defined by
\[
\mathcal{R} (v, \theta) := \frac{\int_{\Omega} v^2_{3, 3} + \frac{1}{\varepsilon^2} (\theta^2)^2 d^3x}{\int_{\Omega} 2 W_{13}^2 (v) + W_{23}^2 (v) + \theta^2 d^3x}, \quad \text{if } (v, \theta) \neq (0, 0) \text{ and } v \in A^\varepsilon \cap A_{BN} =: A_{BN}^\varepsilon,
\]
and $\mathcal{R} (v, \theta) := +\infty$ otherwise. The sequence $\mathcal{R}^\varepsilon$ $\Gamma$-converges to $\mathcal{R}$ in the following sense:

(i) for every sequence $\{ v^\varepsilon \} \subset A^\varepsilon$ and for every $(v, \theta) \in A^\varepsilon \times H^1_{2} (0, \ell)$ such that $v^\varepsilon \rightharpoonup v$ and $(\varepsilon \nabla v^\varepsilon)_{21, 12} \rightharpoonup \theta$, we have that $\mathcal{R} (v, \theta) \leq \liminf_{\varepsilon \to 0} \mathcal{R}^\varepsilon (v^\varepsilon)$;
(ii) for every $(v, \theta) \in A^\varepsilon \times H^1_{2} (0, \ell)$ there exists a sequence $\{ v^\varepsilon \} \subset A^\varepsilon$ such that $v^\varepsilon \rightharpoonup v$, $(\varepsilon \nabla v^\varepsilon)_{21, 12} \rightharpoonup \theta$, and $\limsup_{\varepsilon} \mathcal{R}^\varepsilon (v^\varepsilon) \leq \mathcal{R} (v, \theta)$.

In order to prove the $\liminf$ inequality (i), we use the lower semicontinuity of the numerator of $\mathcal{R}^\varepsilon$ with respect to the weak convergence in $L^2$ of $E^\varepsilon (v^\varepsilon)$, and certain arguments of common use to derive rod theories (see for example [1,11,42]). We also use the strong convergence of $\varepsilon \nabla v^\varepsilon$ in the denominator. To this aim we use the next theorem, where $\mathbb{R}^{3 \times 3}_{\text{skev}}$ is the space of skew-symmetric $3 \times 3$ matrices and $W(u) = \frac{1}{2} (\nabla u - \nabla u^T)$. 

Theorem 2.2. Let \( \{ \mathbf{v}^\varepsilon \} \subset \mathcal{A}_0 \) be such that \( \sup_{\varepsilon} \| \mathbf{E}^\varepsilon (\mathbf{v}^\varepsilon) \|_{L^2} < +\infty \). Then, up to a subsequence, we have
\[
\mathbf{v}^\varepsilon \rightharpoonup^H \mathbf{v} \in \mathcal{A}_0, \quad \mathbf{E}^\varepsilon (\mathbf{v}) \rightharpoonup^L \mathbf{E} \quad \text{with} \quad E_3(\mathbf{v}) = 0 \quad \text{and} \quad E_{33}(\mathbf{v}) = E_{33},
\]
\[
\varepsilon \nabla \mathbf{v}^\varepsilon \rightharpoonup^L \mathbf{W} \in H^1(\Omega; \mathbb{R}^{3 \times 3}_{\text{skew}}) \quad \text{with} \quad W_{\alpha 3} = W_{\alpha 3}(\mathbf{v}).
\]
Moreover, there exist \( \theta \in H^1_\sharp (0, \ell) \) and \( \varphi \in L^2(0, \ell; H^1(\omega)) \) such that
\[
W_{21}(\mathbf{x}) = \theta(x_3), \quad 2E_{13}(\mathbf{x}) = \varphi,_{1}(x_1, x_2) - x_{3} \theta'(x_3), \quad 2E_{23}(\mathbf{x}) = \varphi,_{2}(x_1, x_2) + x_{1} \theta'(x_3).
\]

By a standard result from \( \Gamma \)-convergence, see [6], Theorem 2.1 and Theorem 2.2 imply that
\[
\lim_{\varepsilon \to 0} \inf_{\mathbf{v} \in \mathcal{A}_0} \mathcal{R}^\varepsilon (\mathbf{v}) = \min_{(\mathbf{v}, \theta) \in \mathcal{A}_1^{BN} \times H^1_\sharp (0, \ell)} \mathcal{R}(\mathbf{v}, \theta).
\]

It is shown in [14] that the Bernoulli–Navier space \( \mathcal{A}_1^{BN} \) defined in the statement of Theorem 2.1 can be characterized as follows
\[
\mathcal{A}_1^{BN} = \{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^3) : \nu_\alpha (\mathbf{x}) = w_\alpha (x_3), \quad \nu_3 (\mathbf{x}) = w_3 (x_3) - x_\alpha w,_{\alpha}(x_3), \quad w_\alpha \in H^2(0, \ell), \quad w_3 \in H^1(0, \ell) \},
\]
where a prime denotes differentiation. From this characterization we derive
\[
\min_{(\mathbf{v}, \theta) \in \mathcal{A}_1^{BN} \times H^1_\sharp (0, \ell)} \mathcal{R}(\mathbf{v}, \theta) = \min_{(\mathbf{v}, \theta) \in \mathcal{A}_1^{BN} \times H^1_\sharp (0, \ell)} \mathcal{R}(\mathbf{v}, \theta).
\]

From (7), by means of standard Poincaré’s inequalities, we arrive at (3). The statements contained in (4) are a direct consequence of the assumption \( \sup_{\varepsilon} \| \mathbf{E}^\varepsilon (\mathbf{v}^\varepsilon) \|_{L^2} < +\infty \). The characterization of \( W_{\alpha 3} \) proved under the assumption that \( \omega \) is simply connected, follows from a compatibility equation between infinitesimal strain and infinitesimal rotation.

The proof of the strong convergence statement (5) is quite delicate and it is achieved in several steps. First the function \( \mathbf{v} \) is extended, by using a method of [17], to the infinite cylinder \( \omega \times (-\infty, +\infty) \) in such a way that \( \| \mathbf{E}^\varepsilon (\mathbf{v}^\varepsilon) \|_{L^2(\omega \times (-\infty, +\infty))} \leq C \| \mathbf{E}^\varepsilon (\mathbf{v}^\varepsilon) \|_{L^2(\Omega)} \). Then, by mollifying \( \varepsilon \nabla \mathbf{v}^\varepsilon \) with respect to \( x_3 \) and by integrating over \( \omega \), a function \( \mathbf{H}^\varepsilon = \mathbf{H}^\varepsilon (x_3) \) is defined. An argument based on the invariance of Korn’s constant under homothetic scaling (see [10,9]) yields a bound on the oscillation of \( \varepsilon \nabla \mathbf{v}^\varepsilon \) which, after appropriate estimates, leads to \( \| (\mathbf{H}^\varepsilon) \|_{L^2(\Omega)} \leq C \| \mathbf{E}^\varepsilon (\mathbf{v}^\varepsilon) \|_{L^2(\Omega)} \) and \( \| \mathbf{H}^\varepsilon - \varepsilon \nabla \mathbf{v}^\varepsilon \|_{L^2} \leq C \| \mathbf{E}^\varepsilon (\mathbf{v}^\varepsilon) \|_{L^2} \to 0 \). From these estimates we deduce that, up to a subsequence, \( \mathbf{H}^\varepsilon \rightharpoonup^H \mathbf{W} \) and that \( \mathbf{W} \) is also the strong \( L^2 \)-limit of \( \varepsilon \nabla \mathbf{v}^\varepsilon \).

The detailed proofs of the results presented in this Note will be given in a forthcoming paper [20]. The arguments presented can also be used to prove similar results for thin-walled beams [7,8], and for plates [3,19,21–23].

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References