C. R. Acad. Sci. Paris, Ser. I 350 (2012) 807–812

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Algebra

On the genus of a division algebra

Sur le genre d'un corps gauche

Vladimir I. Chernousov [a], Andrei S. Rapinchuk [b], Igor A. Rapinchuk [c]

[a] *Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1, Canada*
[b] *Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA*
[c] *Department of Mathematics, Yale University, New Haven, CT 06520-8283, USA*

A R T I C L E I N F O

Article history:
Received 19 August 2012
Accepted after revision 20 September 2012
Available online 4 October 2012

Presented by Jean-Pierre Serre

A B S T R A C T

We define the genus **gen**(D) of a finite-dimensional central division algebra D over a field K as the set of all classes $[D']$ in the Brauer group $\mathrm{Br}(K)$ that are represented by central division K-algebras D' having the same maximal subfields as D. We give examples where **gen**(D) is reduced to a single element, and other examples where it is finite.

R É S U M É

Nous définissons le genre **gen**(D) d'un corps gauche central D de dimension finie sur un corps K comme l'ensemble des classes $[D']$ dans le groupe de Brauer $\mathrm{Br}(K)$ qui sont représentées par des corps gauches D' de centre K ayant les mêmes sous-corps maximaux que D. Nous donnons des exemples où **gen**(D) est réduit à un seul élément, ainsi que d'autres où **gen**(D) est fini.

Version française abrégée

Soit K un corps et $\mathrm{Br}(K)$ son groupe de Brauer. Pour une algèbre centrale simple A sur K, on note $[A]$ sa classe dans $\mathrm{Br}(K)$. On définit le genre **gen**(D) d'un corps gauche central D sur K comme l'ensemble des classes $[D'] \in \mathrm{Br}(K)$, où D' est un corps gauche central sur K ayant les mêmes sous-corps maximaux que D. Dans cette note, on étudie les deux questions suivantes :

Question 1. Quand est-ce que le genre est réduit à un seul élément ?

Question 2. Quand est-ce que **gen**(D) est fini ?

On observe que **gen**(D) peut être réduit à un seul élément seulement si $[D]$ est d'exposant deux dans $\mathrm{Br}(K)$; en effet, dans cette situation, **gen**(D) consiste d'un seul élément si K est un corps global. On prouve, en particulier, que si K est un corps de car. $\neq 2$ qui a la propriété que $|\mathbf{gen}(D)| = 1$ pour tout corps gauche D sur K d'exposant deux, alors le corps de fractions rationelles $K(x)$ a la même propriété. Par conséquent, $|\mathbf{gen}(D)| = 1$ pour tout corps gauche D d'exposant deux sur $K = k(x_1, \ldots, x_r)$, où k est soit un corps de nombres soit un corps fini de car. $\neq 2$.

E-mail addresses: vladimir@ualberta.ca (V.I. Chernousov), asr3x@virginia.edu (A.S. Rapinchuk), igor.rapinchuk@yale.edu (I.A. Rapinchuk).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.09.014

On établit aussi le résultat de finitude suivant : soit K un corps de type fini sur son sous-corps premier, soit D un corps gauche central sur K de degré n, $(n, \mathrm{car}.\,K) = 1$. Alors $|\mathbf{gen}(D)|$ est fini. La preuve est réduite à la démonstration de la finitude du sous-groupe de n-torsion ${}_n\mathrm{Br}(K)_V$ du groupe de Brauer de K non-ramifié par rapport à un ensemble convenable de valuations discrètes de K. On donne un exemple d'une borne explicite pour $|\mathbf{gen}(D)|$ dans le cas où D est une algèbre de quaternions sur le corps de fractions $k(E)$ d'une courbe elliptique définie sur un corps de nombres k.

1. Introduction

Let K be a field, $\mathrm{Br}(K)$ be its Brauer group, and for any integer $n > 1$ let ${}_n\mathrm{Br}(K)$ be the subgroup of $\mathrm{Br}(K)$ annihilated by n. For a finite-dimensional central simple algebra A over K, we let $[A]$ denote the corresponding class in $\mathrm{Br}(K)$, and we then define the *genus* $\mathbf{gen}(D)$ of a central division K-algebra D of degree n to be the set of classes $[D'] \in \mathrm{Br}(K)$ where D' is a central division K-algebra having the same maximal subfields as D (in more precise terms, this means that D' has the same degree n, and a field extension P/K of degree n admits a K-embedding $P \hookrightarrow D$ if and only if it admits a K-embedding $P \hookrightarrow D'$).[1] One can ask the following two questions about the genus of a central division K-algebra D of degree n:

Question 1. When does $\mathbf{gen}(D)$ consist of a single class?

Question 2. When is $\mathbf{gen}(D)$ finite?

We note that since the opposite algebra D^{op} has the same maximal subfields as D, the genus $\mathbf{gen}(D)$ can reduce to a single element only if $[D^{\mathrm{op}}] = [D]$, i.e. if D has exponent 2 in the Brauer group. On the other hand, as follows from the theorem of Artin–Hasse–Brauer–Noether (AHBN), $\mathbf{gen}(D)$ does reduce to a single element for any algebra D of exponent 2 over a global field K (in which case D is necessarily a quaternion algebra).[2] The following theorem (which for quaternion algebras was established earlier in [16]) expands the class of fields with this property:

Theorem 1 *(Stability Theorem). Let K be a field of characteristic $\neq 2$.*

(1) *If K satisfies the following property*:
  (∗) *if D and D' are central division K-algebras of exponent* 2 *having the same maximal subfields then $D \simeq D'$ (in other words, for any D of exponent* 2, $|\mathbf{gen}(D) \cap {}_2\mathrm{Br}(K)| = 1$),
  then the field of rational functions $K(x)$ also satisfies (∗).
(2) *If* $|\mathbf{gen}(D)| = 1$ *for any central division K-algebra D of exponent* 2, *then the same is true for any central division $K(x)$-algebra of exponent* 2.

Corollary 2. *Let k be either a finite field of characteristic $\neq 2$ or a number field, and $K = k(x_1, \ldots, x_r)$ be a finitely generated purely transcendental extension of k. Then for any central division K-algebra D of exponent* 2 *we have* $|\mathbf{gen}(D)| = 1$.

While Question 1 makes sense only for division algebras of exponent 2, Question 2 can be asked for arbitrary division algebras. As above, it follows from (ABHN) that $\mathbf{gen}(D)$ is finite for any finite-dimensional central division algebra D over a global field K. For fields other than global, the finiteness question was investigated in [10] for the genus $\mathbf{gen}'(D)$ defined in terms of all finite-dimensional splitting fields (note that $\mathbf{gen}'(D) \subset \mathbf{gen}(D)$) for division algebras D of arbitrary prime exponent p over the field $K = k(x)$ of rational functions, with $p \neq \mathrm{char}\,k$. In particular, it was shown in [10] that if $\mathbf{gen}'(\Delta)$ is finite for any central division algebra Δ of exponent p over a field k, then $\mathbf{gen}'(D)$ is finite for any central division algebra D of exponent p over $K = k(x)$. At the same time, a direct generalization of the construction described in [8, §2] enables one to provide an example of a quaternion division algebra D over an infinitely generated field K with infinite genus $\mathbf{gen}(D)$. So, the following finiteness result seems to cover the most general situation:

Theorem 3. *Let K be a finitely generated field* (*i.e., a finitely generated extension of its prime field*). *If D is a central division K-algebra of exponent prime to* $\mathrm{char}\,K$, *then* $\mathbf{gen}(D)$ *is finite.*

2. The genus and the unramified Brauer group

We will now describe a general set-up that allows one to estimate the size of $\mathbf{gen}(D)$, and will then apply it to proving Theorems 1 and 3. Given a discrete valuation v of K, we let $\mathcal{O}_{K,v}$ and $\overline{K}_v$ denote its valuation ring and residue field,

[1] At the end of this note, we will discuss a generalization of this notion to absolutely almost simple algebraic K-groups in which maximal subfields are replaced with maximal K-tori. We observe in this respect that only *separable* maximal subfields of D give rise to maximal K-tori of $G = \mathrm{SL}_{1,D}$. So, in order to make our definitions fully compatible, one should define $\mathbf{gen}(D)$ in terms of maximal separable subfields. In the current note, however, the degree n of D will always be assumed to be coprime to the characteristic of K, so the issue of separability will not arise.

[2] Indeed, (ABHN) implies that a quaternion algebra over a global field is uniquely determined by its set of ramified places; on the other hand, if two quaternion division algebras have the same maximal subfields, they necessarily have the same ramified places.

respectively. Fix an integer $n > 1$ (which will later be either the degree or the exponent of D) and suppose that V is a set of discrete valuations of K that satisfies the following three conditions:

(A) *For any $a \in K^\times$, the set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite*;
(B) *There exists a finite subset $V' \subset V$ such that the field of fractions of*

$$\mathcal{O} := \bigcap_{v \in V \setminus V'} \mathcal{O}_{K,v}$$

coincides with K;
(C) *For any $v \in V$, the characteristic of $\overline{K}_v$ is prime to n.*

(We note that if K is finitely generated, then (B) is an automatic consequence of (A).) Due to (C), for each $v \in V$ one can define the residue map

$$\rho_v : {}_n\mathrm{Br}(K) \to \mathrm{Hom}\big(\mathcal{G}^{(v)}, \mathbb{Z}/n\mathbb{Z}\big),$$

where $\mathcal{G}^{(v)}$ is the absolute Galois group of $\overline{K}_v$ (cf., for example, [17, §10] or [18, Ch. II, Appendix]). As usual, a class $[A] \in {}_n\mathrm{Br}(K)$ (or a central simple K-algebra A representing this class) is said to be *unramified* at v if $\rho_v([A]) = 1$, and *ramified* otherwise. We let $\mathrm{Ram}_V(A)$ (or $\mathrm{Ram}_V([A])$) denote the set of all $v \in V$ where A is ramified.

Proposition 4. *If V satisfies conditions* (A), (B), *and* (C), *then for any $[A] \in {}_n\mathrm{Br}(K)$, the set $\mathrm{Ram}_V([A])$ is finite.*

Proposition 5. *Let D and D' be central division K-algebras such that $[D] \in {}_n\mathrm{Br}(K)$ and $[D'] \in \mathbf{gen}(D) \cap {}_n\mathrm{Br}(K)$. Given $v \in V$, we let χ_v and $\chi'_v \in \mathrm{Hom}(\mathcal{G}^{(v)}, \mathbb{Z}/n\mathbb{Z})$ denote the images under ρ_v of the classes $[D]$ and $[D']$, respectively. Then*

$$\mathrm{Ker}\,\chi_v = \mathrm{Ker}\,\chi'_v$$

for all $v \in V$. In particular, if D is unramified at v then so is D'.

We define the unramified part of ${}_n\mathrm{Br}(K)$ relative to V as follows:

$${}_n\mathrm{Br}(K)_V := \bigcap_{v \in V} \mathrm{Ker}\,\rho_v.$$

The following statement relates the size of the genus to the size of ${}_n\mathrm{Br}(K)_V$:

Theorem 6. *Assume that ${}_n\mathrm{Br}(K)_V$ is finite. Then for any finite-dimensional central division K-algebra D of exponent n, the intersection $\mathbf{gen}(D) \cap {}_n\mathrm{Br}(K)$ is finite, of size*

$$\big|\mathbf{gen}(D) \cap {}_n\mathrm{Br}(K)\big| \leqslant \big|{}_n\mathrm{Br}(K)_V\big| \cdot \varphi(n)^r, \quad \textit{with } r = \big|\mathrm{Ram}_V(D)\big|,$$

where φ is the Euler function. In particular, if D has degree n then

$$\big|\mathbf{gen}(D)\big| \leqslant \big|{}_n\mathrm{Br}(K)_V\big| \cdot \varphi(n)^r.$$

We will now specialize to the situation where $K = k(C)$ is the field of rational functions on a smooth absolutely irreducible projective curve C over a field k. Set V to be the set of all geometric places of K, i.e. those discrete valuations of K that are trivial on k. Then the corresponding unramified Brauer group ${}_n\mathrm{Br}(K)_V$ will be denoted by ${}_n\mathrm{Br}(K)_{\mathrm{ur}}$ (this is precisely the n-torsion subgroup of the Brauer group of the curve C). Applying the techniques outlined above, in conjunction with some considerations involving specialization, we obtain the following:

Theorem 7. *Let $n > 1$ be an integer prime to* $\mathrm{char}\, k$. *Assume that*

- *the set $C(k)$ of rational points is infinite*;
- $|{}_n\mathrm{Br}(K)_{\mathrm{ur}}/\iota_k({}_n\mathrm{Br}(k))| =: M < \infty$, *where $\iota_k : \mathrm{Br}(k) \to \mathrm{Br}(K)$ is the canonical map.*

Then

(1) *if there exists $N < \infty$ such that*

$$\big|\mathbf{gen}(\Delta) \cap {}_n\mathrm{Br}(k)\big| \leqslant N$$

for any central division k-algebra Δ of exponent n, then for any central division K-algebra D of exponent n we have

$$\left|\mathbf{gen}(D) \cap {}_n\mathrm{Br}(K)\right| \leqslant M \cdot N \cdot \varphi(n)^r,$$

where $r = |\mathrm{Ram}_V(D)|$;

(2) *if $\mathbf{gen}(\Delta) \cap {}_n\mathrm{Br}(k)$ is finite for any central division k-algebra Δ of exponent n, then $\mathbf{gen}(D) \cap {}_n\mathrm{Br}(K)$ is finite for any central division K-algebra D of exponent n.*

One notable case where Theorem 7 applies is $C = \mathbb{P}^1_k$ over an infinite field k (which we can assume without loss of generality). It is well-known that in this case ${}_n\mathrm{Br}(K)_{\mathrm{ur}} = \iota_k({}_n\mathrm{Br}(k))$ (cf. [9, Corollary 6.4.6]), i.e. one can take $M = 1$. Now, let $n = 2$ and assume that k satisfies condition $(*)$ of Theorem 1, i.e. $|\mathbf{gen}(\Delta) \cap {}_2\mathrm{Br}(k)| = 1$ for any central division k-algebra Δ of exponent 2. The latter means that one can take $N = 1$. We then obtain from Theorem 7 that $|\mathbf{gen}(D) \cap {}_2\mathrm{Br}(K)| = 1$ for any central division K-algebra D of exponent 2, proving part (1) of Theorem 1. The proof of part (2) is similar.

Furthermore, it follows from Theorem 6 that in order to prove Theorem 3, it is enough to establish the following:

Theorem 8. *Let K be a finitely generated field, and let $n > 1$ be an integer coprime to $\mathrm{char}\, K$. Then there exists a set V of discrete valuations of K that satisfies conditions (A), (B) and (C), and for which the unramified Brauer group ${}_n\mathrm{Br}(K)_V$ is finite.*

We originally proved Theorem 8 by a method related to the proof of the Weak Mordell–Weil Theorem (cf. [11, Ch. VI]), which in principle can be used to obtain some estimates on the size of ${}_n\mathrm{Br}(K)_V$, hence of $\mathbf{gen}(D)$ (see below). It was later pointed out to us by J.-L. Colliot-Thélène [4] that a (nonconstructive) proof of the finiteness of ${}_n\mathrm{Br}(K)_V$ can be derived from the following general statement:

Theorem 9. *Let X be a scheme of finite type over $U = \mathrm{Spec}\, A$, where A is either a finite field or the ring of S-integers in a number field (with S finite). For any integer n invertible in A and any n-torsion constructible sheaf $\mathfrak{F}$ on X, the étale cohomology groups $H^i_{\text{ét}}(X, \mathfrak{F})$ are finite for all $i \geqslant 0$.*

Given a finitely generated field K and an integer $n > 1$ prime to $\mathrm{char}\, K$, we can pick a smooth affine integral scheme X as in Theorem 9 with the field of rational functions K. Applying Theorem 9 to the étale sheaf associated with the group scheme $\boldsymbol{\mu}_n$ of nth roots of unity, we obtain the finiteness of $H^2_{\text{ét}}(X, \boldsymbol{\mu}_n)$. Then the Kummer sequence yields the finiteness of ${}_n\mathrm{Br}(X)$. On the other hand, it follows from the absolute purity conjecture proved by O. Gabber (see [7] for an exposition of Gabber's proof, and also [5, p. 153] and [3, discussion after Theorem 4.2] regarding the history of the question) that the latter coincides with ${}_n\mathrm{Br}(K)_V$, where V is the set of discrete valuations of K associated with the divisors of X, cf. [7], hence the required fact (obviously, this V satisfies our conditions (A), (B) and (C)).

Since the proof of Theorem 9 is not readily available in the existing literature, we reproduce below an outline of the argument kindly explained to us by J.-L. Colliot-Thélène in [4] (with his permission). Since for our purposes we only need to consider the smooth case, in the situation where A is a finite field the required fact follows from Corollary 4.5 or Corollary 5.5 in [12, Ch. VI] in conjunction with the Hochschild–Serre spectral sequence (cf. [12, Ch. III, Theorem 2.20]).

Let now A be a ring of S-integers in some number field k, where S is a finite set of places of k. Applying to the structure morphism $f : X \to U$ Theorem 1.1 of the chapter "Théorèmes de finitude" in Deligne's book [6, p. 233], we obtain that the direct images $\mathrm{R}^q f_* \mathfrak{F}$ are constructible n-torsion sheaves on U. Combining Proposition 2.9 in [13, Ch. II] with Theorem 8.3.19 in [14], we obtain that the groups $H^p_{\text{ét}}(U, \mathrm{R}^q f_* \mathfrak{F})$ are finite for all $p \geqslant 0$. Then the Leray spectral sequence $H^p_{\text{ét}}(U, \mathrm{R}^q f_* \mathfrak{F}) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathfrak{F})$ [12, Ch. III, Theorem 1.18] shows that the groups $H^i_{\text{ét}}(X, \mathfrak{F})$ are all finite.

3. An example

We will now show how the methods involved in our original proof of Theorem 8 can actually be used to estimate the size of the unramified Brauer group, and hence of the genus of a division algebra, in certain situations. Because of space limitation, we will focus on the following example. Let k be a number field, and let E be an elliptic curve over k given by a Weierstrass equation

$$y^2 = f(x) \quad \text{where } f(x) = x^3 + \alpha x^2 + \beta x + \gamma.$$

Without loss of generality, we may assume that all the coefficients lie in the ring of integers $\mathcal{O}_k$. We will also assume that E splits over k, i.e. f has three roots in k. Let $\delta \neq 0$ be the discriminant of f, and set

$$S = V^k_\infty \cup V^k(2) \cup V^k(\delta)$$

where V^k denotes the set of all valuations of k, V^k_∞ the subset of archimedean valuations, and for $a \in k^\times$ we set $V^k(a) = \{v \in V^k \setminus V^k_\infty \mid v(a) \neq 0\}$. Let

$$K := k(E) = k(x, y).$$

For a nonarchimedean $v \in V^k$, let $\tilde{v}$ denote its extension to $F := k(y)$ given by

$$\tilde{v}\big(a_m y^m + \cdots + a_0\big) = \min_{a_i \neq 0} v(a_i)$$

(cf. [1, Ch. VI, §10]). It can be shown that for $v \in V^k \setminus S$, the valuation $\tilde{v}$ has a unique extension to K, which we will denote by $w = w(v)$. We now introduce the following set of discrete valuations of K:

$$V = V_0 \cup V_1,$$

where V_0 is the set of all geometric places of K (i.e., those discrete valuations that are trivial on k), and V_1 consists of the valuations $w = w(v)$ for all $v \in V^k \setminus S$. It is easy to see that V satisfies conditions (A), (B) and (C).

Theorem 10. *The unramified Brauer group ${}_2\mathrm{Br}(K)_V$ is finite of order dividing*

$$2^{|S|-t} \cdot \big|{}_2\mathrm{Cl}_S(k)\big|^2 \cdot \big|U_S(k)/U_S(k)^2\big|^2,$$

where $t = c + 1$ with c being the number of complex places of k, and $\mathrm{Cl}_S(k)$ and $U_S(k)$ are the class group and the group of units of the ring of S-integers $\mathcal{O}_k(S)$, respectively.

Sketch of proof. We will use the following description of the 2-torsion ${}_2\mathrm{Br}(K)_{V_0}$ in the geometric Brauer group [2]: *If E splits over k, i.e. $f(x) = (x-a)(x-b)(x-c)$ with $a, b, c \in k$, then ${}_2\mathrm{Br}(K)_{V_0} = {}_2\mathrm{Br}(k) \oplus I$, where $I \subset {}_2\mathrm{Br}(K)_{V_0}$ is a subgroup such that every element of I is represented by a bi-quaternion algebra $(r, x-b)_K \otimes_K (s, x-c)_K$ for some $r, s \in k^\times$.* Let $[D] \in {}_2\mathrm{Br}(K)_V$. Then $[D] = [\Delta' \otimes_K \Delta'']$ where $\Delta' = \Delta_0 \otimes_k K$ for some central division k-algebra Δ_0 of exponent 2, and $\Delta'' = (r, x-b)_K \otimes_K (s, x-c)_K$ for some $r, s \in k^\times$. Using the corestriction map $\mathrm{cor}_{K/F}$, one shows that Δ_0 is unramified at all $v \in V^k \setminus S$, and hence Δ'' is unramified at all $w \in V_1$. The latter implies that $v(r), v(s) \equiv 0 \pmod 2$ for all $v \in V^k \setminus S$. Let

$$\tilde{\Gamma} = \big\{x \in k^\times \;\big|\; v(x) \equiv 0 \ (\mathrm{mod}\ 2) \text{ for all } v \in V^k \setminus S\big\},$$

and let Γ be the image of $\tilde{\Gamma}$ in $k^\times/k^{\times 2}$. Then there is an exact sequence

$$0 \to U_S(k)/U_S(k)^2 \to \Gamma \to {}_2\mathrm{Cl}_S(k) \to 0$$

(cf. [11, §6.1]), hence $|\Gamma| = |{}_2\mathrm{Cl}_S(k)| \cdot |U_S(k)/U_S(k)^2|$.

Our previous discussion shows that there are at most $|\Gamma|^2$ possibilities for Δ''. On the other hand, it follows from (ABHN) that ${}_2\mathrm{Br}(k)_{V^k \setminus S}$ has order $2^{|S|-t}$, which bounds the number of possibilities for Δ'. Combining this with the above computation of $|\Gamma|$, we obtain our claim. □

Example. Consider an elliptic curve E over $\mathbb{Q}$ given by $y^2 = x^3 - x$. We have $\delta = 4$, so $S = \{\infty, 2\}$. Furthermore,

$$|S| - t = 1, \qquad \mathrm{Cl}_S(\mathbb{Q}) = 1 \quad \text{and} \quad U_S(\mathbb{Q}) = \{\pm 1\} \times \mathbb{Z}.$$

So, by Theorem 10, for $K = \mathbb{Q}(E)$ and the set V constructed above, the group ${}_2\mathrm{Br}(K)_V$ has order dividing $2 \cdot 4^2 = 32$. Combining this with Theorem 6, we obtain that for any quaternion algebra D over K, we have $|\mathbf{gen}(D)| \leqslant 32$.

4. Concluding remarks

The questions considered in this note for division algebras can be analyzed in the broader context of arbitrary absolutely almost simple simply connected (or adjoint) K-groups. In this set-up, one can define the genus of such a K-group G as the collection of K-forms G' of G that have the same isomorphism classes of maximal K-tori (as a variation, one can base the notion only on *generic* tori). We note that questions about groups in the same genus arise in the analysis of weak commensurability of Zariski-dense subgroups which in turn is related to some problems in differential geometry, cf. [15]. In view of our Theorem 3, it seems natural to propose the following:

Conjecture. *Let G be an absolutely almost simple simply connected algebraic group over a finitely generated field K of characteristic zero (or of "good" characteristic relative to G). Then there exists a* finite *collection $G_1, \ldots, G_r$ of K-forms of G such that if H is a K-form of G having the same isomorphism classes of maximal K-tori as G, then H is K-isomorphic to one of the G_i's.*

Our proof of Theorem 3 yields in fact a proof of this conjecture for inner forms of type A_ℓ.

Theorem 11. *Let G be an absolutely almost simple simply connected algebraic group of inner type A_ℓ over a finitely generated field K whose characteristic is either zero or does not divide $\ell + 1$. Then the above conjecture is true for G.*

(In this regard, we note that if central simple K-algebras $A_1 = M_{\ell_1}(D_1)$ and $A_2 = M_{\ell_2}(D_2)$, where D_1 and D_2 are division algebras, have the same maximal étale K-subalgebras, then $\ell_1 = \ell_2$ and D_1 and D_2 have the same maximal separable subfields, cf. [16, Lemma 2.3].)

We plan to address the general case of the conjecture in our subsequent publications.

Acknowledgements

We are grateful to J.-L. Colliot-Thélène for the correspondence regarding Theorems 8 and 9, and for his permission to include his proof of Theorem 9. Thanks are also due to the anonymous referee for his/her comments. The first-named author was partially supported by the Canada Research Chairs Program and an NSERC research grant. The second-named author was partially supported by NSF grant DMS-0965758, BSF grant 2010149, and the Humboldt Foundation.

References

[1] N. Bourbaki, Commutative Algebra, Springer, 1989 (Chapters 1–7).
[2] V. Chernousov, V. Guletskii, 2-Torsion of the Brauer group of an elliptic curve, Doc. Math. Extra volume (2001) 85–120.
[3] J.-L. Colliot-Thélène, Groupes de Chow des zéro-cycles sur les variétés p-adiques [d'après S. Saito, K. Sato et al.], in: Séminaire Bourbaki 2009–2010, Astérisque 339 (2011) 1–30, Exp. 1012.
[4] J.-L. Colliot-Thélène, Private communication, 2012.
[5] J.-L. Colliot-Thélène, S. Saito, Zéro-cycles sur les variétés p-adiques et groupe de Brauer, Int. Math. Res. Not. IMRN 4 (1996) 151–160.
[6] P. Deligne, Cohomologie étale, SGA $4\frac{1}{2}$, Lecture Notes in Math., vol. 569, Springer, 1977.
[7] K. Fujiwara, A proof of the absolute purity conjecture (after Gabber), in: Algebraic Geometry 2000, Azumino (Hotaka), in: Adv. Stud. Pure Math., vol. 36, Math. Soc. Japan, 2002, pp. 153–183.
[8] S. Garibaldi, D. Saltman, Quaternion algebras with the same subfields, in: Quadratic Forms, Linear Algebraic Groups, and Cohomology, in: Dev. Math., vol. 18, Springer, New York, 2010, pp. 225–238.
[9] P. Gille, T. Szamuely, Central Simple Algebras and Galois Cohomology, Cambridge Univ. Press, 2006.
[10] D. Krashen, K. McKinnie, Distinguishing division algebras by finite splitting fields, Manuscripta Math. 134 (2011) 171–182.
[11] S. Lang, Fundamentals of Diophantine Geometry, Springer, 2010.
[12] J.S. Milne, Étale Cohomology, Princeton Univ. Press, 1980.
[13] J.S. Milne, Arithmetic Duality Theorems, second ed., Kea Books, 2006.
[14] J. Neukirch, A. Schmidt, K. Wingberg, Cohomology of Number Fields, Grundlehren Math. Wiss., vol. 323, Springer, 2000.
[15] G. Prasad, A.S. Rapinchuk, Weakly commensurable arithmetic groups and isospectral locally symmetric spaces, Publ. Math. Inst. Hautes Etudes Sci. 109 (2009) 113–184.
[16] A.S. Rapinchuk, I.A. Rapinchuk, On division algebras having the same maximal subfields, Manuscripta Math. 132 (2010) 273–293.
[17] D. Saltman, Lectures on Division Algebras, CBMS Reg. Conf. Ser., vol. 94, Amer. Math. Soc., 1999.
[18] J.-P. Serre, Galois Cohomology, Springer, 1997.