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Differential Geometry

# Hebey–Vaugon conjecture II

## *La conjecture de Hebey–Vaugon II*

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ABSTRACT

In this Note, we consider the remaining cases of Hebey–Vaugon conjecture. Assuming the positive mass theorem, we give a positive answer to this conjecture.



RÉSUMÉ

Dans cette Note, on considère les cas restants de la conjecture de Hebey–Vaugon. En admettant la théorème de la masse positive, on donne une réponse positive à cette conjecture.



Let $(M,g)$ be a compact Riemannian manifold of dimension $n \geqslant 3$. Denote by $I(M,g)$, $C(M,g)$ and $R_g$ the isometry group, the conformal transformations group and the scalar curvature, respectively. Let $G$ be a subgroup of the isometry group $I(M,g)$. The equivariant Yamabe problem can be formulated as follows: *in the conformal class of g, there exists a G-invariant metric with constant scalar curvature*. Assuming the positive mass theorem and the Weyl vanishing conjecture (for more details on the subject, see [5,10] and the references therein), E. Hebey and M. Vaugon [4] proved that this problem has solutions. Moreover, they proved that the infimum of Yamabe functional

$$I_g(\varphi) = \left( \int_M |\nabla\varphi|^2 + \frac{n-2}{4(n-1)} R_g \varphi^2 \,\mathrm{d}v \right) \|\varphi\|^{-2}_{\frac{2n}{n-2}}, \tag{1}$$

over $G$-invariant nonnegative functions is achieved by a smooth positive $G$-invariant function. This function is a solution of the Yamabe equation, which is the Euler–Lagrange equation of $I_g$:

$$\Delta_g \varphi + \frac{n-2}{4(n-1)} R_g \varphi = \mu \varphi^{\frac{n+2}{n-2}}.$$

One of the consequences of these results is that the following conjecture due to Lichnerowicz [7] is true:

**Lichnerowicz conjecture.** *For every compact Riemannian manifold $(M,g)$ which is not conformal to the unit sphere $S^n$ endowed with its standard metric $g_s$, there exists a metric $\tilde{g}$ conformal to $g$ for which $I(M,\tilde{g}) = C(M,g)$, and the scalar curvature $R_{\tilde{g}}$ is constant.*

The classical Yamabe problem, which consists of finding a conformal metric with constant scalar curvature on a compact Riemannian manifold, is a particular case of the equivariant Yamabe problem (it corresponds to $G = \{\mathrm{id}\}$). This problem

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was completely solved by H. Yamabe [13], N. Trudinger [12], T. Aubin [1] and R. Schoen [11]. The main idea to prove the existence of positive minimizers for $I_g$ is to show that if $(M, g)$ is not conformal to the sphere endowed with its standard metric, then

$$\mu(g) := \inf_{C^\infty(M)} I_g(\varphi) < \frac{1}{4}n(n-2)\omega_n^{2/n}, \tag{2}$$

where $\omega_n$ is the volume of the unit sphere $S^n$.

T. Aubin [1] proved (2) in some cases by constructing a test function $u_\varepsilon$ satisfying $I_g(u_\varepsilon) < \frac{1}{4}n(n-2)\omega_n^{2/n}$. He conjectured that (2) always holds except for the sphere. R. Schoen constructed another test function which involves the Green function of the conformal Laplacian $\Delta_g + \frac{n-2}{4(n-1)}R_g$. Using the positive mass theorem, R. Schoen proved (2) for all compact manifolds which are not conformal to $(S^n, g_s)$. The solution of the Yamabe problem follows.

Later, E. Hebey and M. Vaugon [4] showed that we can generalize (2) for the equivariant case as follows:

Denote by $O_G(P)$ the orbit of $P \in M$ under $G$ and by $\operatorname{card} O_G(P)$ its cardinality. Let $C_G^\infty(M)$ be the set of smooth $G$-invariant functions and

$$\mu_G(g) := \inf_{C_G^\infty(M)} I_g(\varphi).$$

Following E. Hebey and M. Vaugon [3,4], we define the integer $\omega(P)$ at a point $P$ as

$$\omega(P) = \inf\{i \in \mathbb{N} / \|\nabla^i W_g(P)\| \neq 0\} \quad \big(\omega(P) = +\infty \text{ if } \forall i \in \mathbb{N},\ \|\nabla^i W_g(P)\| = 0\big).$$

Hebey–Vaugon conjecture. *Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geqslant 3$ and $G$ be a subgroup of $I(M, g)$. If $(M, g)$ is not conformal to $(S^n, g_s)$ or if the action of $G$ has no fixed point, then the following inequality holds*

$$\mu_G(g) < \frac{1}{4}n(n-2)\omega_n^{2/n}\Big(\inf_{Q\in M} \operatorname{card} O_G(Q)\Big)^{2/n}. \tag{3}$$

E. Hebey and M. Vaugon showed that if this conjecture holds, then it implies that the equivariant Yamabe problem has minimizing solutions and the Lichnerowicz conjecture is also true. Notice that if $G = \{\mathrm{id}\}$, then this conjecture corresponds to (2).

Assuming the positive mass theorem, E. Hebey and M. Vaugon [4] proved the following:

**Theorem 1** *(E. Hebey and M. Vaugon). The Hebey–Vaugon conjecture holds if at least one of the following conditions is satisfied*:

1. *The action of $G$ on $M$ is free.*
2. $3 \leqslant \dim M \leqslant 11$.
3. *There exists a point $P \in M$ with finite minimal orbit under $G$ such that $\omega(P) > (n-6)/2$ or $\omega(P) \in \{0, 1, 2\}$.*

The main result of this note is the following:

**Theorem 2.** *If there exists a point $P \in M$ such that $\omega(P) \leqslant (n-6)/2$, then*

$$\mu_G(g) < \frac{1}{4}n(n-2)\omega_n^{2/n}\big(\operatorname{card} O_G(P)\big)^{2/n}. \tag{4}$$

Note that if we assume the positive mass theorem, then Theorem 1 and Theorem 2 implies that the Hebey–Vaugon conjecture holds. In particular, it holds if $M$ is a spin manifold.

The proof of Theorem 2 doesn't require the positive mass theorem. If $\operatorname{card} O_G(Q) = +\infty$ for all $Q \in M$, then (3) holds. So we have to consider only the case when there exists a point in $M$ with finite orbit. From now on, we suppose that $P \in M$ is contained in a finite orbit and $\omega(P) \leqslant \frac{n-6}{2}$. The assumption $\omega(P) \leqslant \frac{n-6}{2}$ deletes the case $(M, g)$ is conformal to $(S^n, g_s)$.

In order to prove Theorem 2, we construct from the function $\varphi_{\varepsilon,P}$ defined below a $G$-invariant test function $\phi_\varepsilon$ such that

$$I_g(\phi_\varepsilon) < \frac{1}{4}n(n-2)\omega_n^{2/n}\big(\operatorname{card} O_G(P)\big)^{2/n}. \tag{5}$$

Let us recall the construction in [9] of $\varphi_{\varepsilon,P}$. Let $\{x^j\}$ be the geodesic normal coordinates in the neighborhood of $P$ and define $r = |x|$ and $\xi^j = x^j/r$. Without loss of generality, we suppose that $\det g = 1 + O(r^N)$, with $N > 0$ sufficiently large (for the existence of such coordinates for a $G$-invariant conformal class, see [4,6]).

$$\varphi_{\varepsilon,P}(Q) = \begin{cases} (1 - r^{\omega(P)+2} f(\xi))\big[\big(\frac{\varepsilon}{r^2+\varepsilon^2}\big)^{\frac{n-2}{2}} - \big(\frac{\varepsilon}{\delta^2+\varepsilon^2}\big)^{\frac{n-2}{2}}\big] & \text{if } Q \in B_P(\delta); \\ 0 & \text{if } Q \in M - B_P(\delta), \end{cases}$$

where $r = d(Q, P)$ is the distance between $P$ and $Q$, and $B_P(\delta)$ is the geodesic ball of center $P$ and radius $\delta$ fixed sufficiently small. $f$ is a function depending only on $\xi$ (defined on $S^{n-1}$), chosen such that $\int_{S^{n-1}} f \, d\sigma = 0$.

Let $\bar{R}$ be the leading part in the Taylor expansion of the scalar curvature $R_g$ in a neighborhood of $P$ and $\mu(P)$ is its degree. Hence,

$$R_g(Q) = \bar{R} + O\left(r^{\mu(P)+1}\right) \quad \text{and} \quad \bar{R} = r^{\mu(P)} \sum_{|\beta|=\mu(P)} \nabla_\beta R_g(P) \xi^\beta.$$

We summarize some properties of $\bar{R}$ in the following proposition:

**Proposition 3.**

1. *$\bar{R}$ is a homogeneous polynomial of degree $\mu(P)$ and is invariant under the action of the stabilizer group of $P$.*
2. *We always have $\mu(P) \geqslant \omega(P)$.*
3. *If $\mu(P) \geqslant \omega(P) + 1$, then $\int_{S^{n-1}(r)} R \, d\sigma < 0$ for $r > 0$ sufficiently small.*
4. *If $\mu(P) = \omega(P)$, then there exist eigenfunctions $\varphi_k$ of the Laplacian on $S^{n-1}$ such that the restriction of $\bar{R}$ to the sphere is given by $\bar{R}|_{S^{n-1}} = \sum_{k=1}^{q} \nu_k \varphi_k$, where $q \leqslant [\omega(P)/2]$, $\Delta_s \varphi_k = \nu_k \varphi_k$ and $\nu_k = (\omega - 2k + 2)(n + \omega - 2k)$ are the eigenvalues of $\Delta_s$ with respect to the standard metric $g_s$ of $S^{n-1}$.*

**Sketch of proof.** Since the scalar curvature is invariant under the action of the isometry group $I(M, g)$, $\bar{R}$ is invariant under the action of the stabilizer of $P$. The second statement of Proposition 3 holds, since $\nabla^i W_g(P) = 0$ for all $i < \omega(P)$ and $\det g = 1 + O(r^N)$ (a complete proof is given in [4], Section 8). The third statement is proven by T. Aubin ([2], Section 3). Using the fact that $\bar{R}$ is homogeneous polynomial of degree $\omega(P)$ and the fact that for all $j \leqslant \omega(P) - 1$

$$\left|\nabla^j R_g(P)\right| = 0, \quad \Delta_g^{j+1} R_g(P) = 0 \quad \text{and} \quad \left|\nabla \Delta_g^{j+1} R_g(P)\right| = 0 \tag{6}$$

(the proof of (6) is given in [4], Section 8), it is proven in [2] that this implies $\Delta_{\mathcal{E}}^{[\omega(P)/2]} \bar{R} = 0$, where $\Delta_{\mathcal{E}}$ is the Euclidean Laplacian. Hence, if we restrict $\bar{R}$ to the sphere, we get the decomposition of the item 4 in Proposition 3. □

Using the split of $\bar{R}$ given in Proposition 3, we proved in [9] that if the cardinality of $O_G(P)$ is minimal and $\omega(P) \leqslant 15$, then there exists $c \in \mathbb{R}$ such that for $f = c\bar{R}|_{S^{n-1}}$, the function $\phi_\varepsilon = \sum_{P_i \in O_G(P)} \varphi_{\varepsilon, P_i}$ is $G$-invariant and satisfies (5). Moreover, we proved the following theorem:

**Theorem 4.** *If $\omega(P) \leqslant (n-6)/2$, then there exist $c_k \in \mathbb{R}$, such that for $f = \sum_{k=1}^{q} c_k \varphi_k$, the function $\varphi_{\varepsilon,P}$ satisfies $I_g(\varphi_{\varepsilon,P}) < \frac{1}{4} n(n-2)\omega_n^{2/n}$.*

The proof of Theorem 4 is technical and uses Proposition 3. It is given in [9] (see also [8] for a detailed proof). Below, we show that using Theorem 4, we can construct a $G$-invariant function $\phi_\varepsilon$ which satisfies (5) for $\omega(P) \leqslant \frac{n-6}{2}$ (the cardinality of $O_G(P)$ is not necessarily minimal). This implies Theorem 2.

**Proof of Theorem 2.** If $\mu(P) \geqslant \omega(P) + 1$, then $\int_{S^{n-1}(r)} R \, d\sigma < 0$ for $r > 0$ sufficiently small (see Proposition 3). The conjecture holds immediately, by choosing $f = 0$, $\varphi_{\varepsilon,P} = u_{\varepsilon,P}$ (see [8,9] for more details).

From now on, we suppose that $\mu(P) = \omega(P)$. Let $H \subset G$ be the stabilizer of $P$. We consider the function $f = \sum_{k=1}^{q} c_k \varphi_k$ of Theorem 4. Using the exponential map on $P$ as a local chart, we can view $f$ and $\varphi_k$ as functions defined over the unit sphere of $T_P M$, the tangent space of $M$ on $P$. Let $h$ be an isometry in $H$ and $h_*(P) : (T_P M, g_P) \to (T_P M, g_P)$ be the linear tangent map of $h$ on $P$. It is a linear isometry with respect to the inner product $g_P$ which is Euclidean. $h_*(P)$ conserves the unit sphere $S^{n-1} \subset T_P M$ and the Laplacian. We already know that the function $\bar{R} = r^{\omega(P)} \sum_{k=1}^{q} \nu_k \varphi_k$ is $H$-invariant. Notice that $\varphi_k$ and $\varphi_j$ belong to two different eigenspaces if $k \neq j$. Since, isometries conserve the Laplacian and $\varphi_k$ are eigenfunctions of the Laplacian on the sphere endowed with its standard metric, it yields that $\varphi_k$ and $f$ are $H$-invariant. On the other hand, we have the following bijective map:

$$\begin{aligned} G/H &\longrightarrow O_G(P) \\ \sigma H &\longmapsto \sigma(P). \end{aligned}$$

Since $f$ is $H$-invariant, $\varphi_{\varepsilon,P}$ is $H$-invariant and the function $\phi_\varepsilon = \sum_{\sigma \in G/H} \varphi_{\varepsilon,P} \circ \sigma^{-1}$ is $G$-invariant and satisfies (5). □

## References

[1] T. Aubin, Équations différentielles non linéaires et problème de Yamabe, J. Math. Pures et Appl. 55 (1976) 269–296.
[2] T. Aubin, Sur quelques problèmes de courbure scalaire, J. Funct. Anal. 240 (2006) 269–289.

[3] E. Hebey, M. Vaugon, Courbure scalaire prescrite pour des variétés non conformément difféomorphes à la sphère, C. R. Acad. Sci. Paris, Ser. I 316 (3) (1993) 281–282.
[4] E. Hebey, M. Vaugon, Le problème de Yamabe équivariant, Bull. Sci. Math. 117 (1993) 241–286.
[5] M. Khuri, F. Marques, R. Schoen, A compactness theorem for the Yamabe problem, J. Diff. Geom. 81 (2009) 143–196.
[6] J.M. Lee, T. Parker, The Yamabe problem, Bull. Amer. Math. Soc. 17 (1987) 37–91.
[7] A. Lichnerowicz, Sur les transformations conformes d'une variété riemannienne compacte, C. R. Acad. Sci. Paris 259 (1964).
[8] F. Madani, Le problème de Yamabe equivariant et la conjecture de Hebey–Vaugon, Ph.D. thesis, Université Pierre et Marie Curie, 2009.
[9] F. Madani, Equivariant Yamabe problem and Hebey–Vaugon conjecture, J. Func. Anal. 258 (2010) 241–254.
[10] F. Marques, Blow-up examples for the Yamabe problem, Calc. Var. 36 (2009) 377–397.
[11] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Differ. Geom. 20 (1984) 479–495.
[12] N. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa 22 (1968) 265–274.
[13] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. 12 (1960) 21–37.