C. R. Acad. Sci. Paris, Ser. I 350 (2012) 827–829

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Complex Analysis

A note on the Bergman kernel of a certain Hartogs domain

Une note sur le noyau de Bergman pour un certain domaine de Hartogs

Atsushi Yamamori

Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, South Korea

ARTICLE INFO

Article history:
Received 13 September 2012
Accepted 8 October 2012
Available online 22 October 2012

Presented by the Editorial Board

ABSTRACT

We give an explicit formula of the Bergman kernel of a certain Hartogs domain.

RÉSUMÉ

Nous obtenons une formule explicite du noyau de Bergman pour un certain domaine de Hartogs.

1. Introduction

After the discovery of the Bergman kernel, many mathematicians tried to find a complex domain with explicit Bergman kernel. However, except for some special cases, it is hard to express the Bergman kernel of a given domain in explicit form. Thus it is fundamental and important to find a domain with explicit Bergman kernel. In this note, we consider the Bergman kernel of the Hartogs domain:

$$\Omega = \{(z, \zeta) \in D \times \mathbb{C}^m;\ \|\zeta\|^2 < p(z)\}.$$

Here p is a positive continuous function on D and the domain $D \subset \mathbb{C}^n$ is called the base domain of the Hartogs domain Ω. If the base domain D is an irreducible bounded symmetric domain or $\mathbb{C}^n$, then there is an explicit formula of the Bergman kernel for a certain p (see [5] and [6]). The main result of this note gives us a new example of the Hartogs domain with explicit Bergman kernel.

Let $F(t)$ be a non-increasing continuous function from an interval $(0, B]$ into $(0, +\infty]$. Further we assume that F extends to a meromorphic function on $B\mathbb{D} = \{z \in \mathbb{C};\ |z| < B\}$. In this note we consider the domain of the form

$$\Omega_{D_F,m} = \{(z, \zeta) \in D_F \times \mathbb{C}^m;\ |\zeta|^2 < (F(|z_1|^2) - |z_2|^2)^s\},$$

where $s > 0$ and the domain D_F is defined by $D_F = \{z \in \mathbb{C}^2;\ |z_1|^2 < B,\ |z_2|^2 < F(|z_1|^2)\}$.

Put

$$a(z, z') = F(z_1\overline{z_1'}) - z_2\overline{z_2'}, \qquad G(t) = -\left(\frac{tF'}{F}\right)',$$

E-mail address: ats.yamamori@gmail.com.

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.10.009

$$g(z, z') = \frac{F(z_1\overline{z_1'})^2}{a(z, z')^3} \cdot G(z_1\overline{z_1'}), \qquad c_k(F^m) = \int_0^B t^k F(t)^m \,dt.$$

In the following, we assume that there exists a real number γ such that

$$\sum_{k=0}^{\infty} t^k / c_k(F^\alpha) = (\alpha + 1 + \gamma)F(t)^{-\alpha}G(t), \quad \text{for any } \alpha \in \mathbb{R}_{>0}. \tag{1}$$

Throughout this note we also assume the following condition:

$$a(z, z)a(z', z') < |a(z, z')|^2, \quad \text{for all } z, z' \in D_F. \tag{2}$$

The weighted Bergman kernel K_α of $L_a^2(D_F, a(z,z)^\alpha \,dz)$ was computed by M. Engliš [2, Corollary 4.14].

Theorem 1.1. *Let F be such that D_F is a complete Reinhardt domain and assume that the condition (1) holds. Then the weighted Bergman kernel K_α is given by*

$$K_\alpha(z, z') = (\alpha + 1)\big[(\alpha + 2) + \gamma\big(1 - w(z, z')\big)\big]a(z, z')^{-\alpha} g(z, z'),$$

where $w(z, z') = z_2\overline{z_2'}/F(z_1\overline{z_1'})$.

In the next section, we will see that this theorem and the Forelli–Rudin construction allow us to compute the Bergman kernel of the domain $\Omega_{D_F,m}$.

2. Bergman kernel

Since our formula is expressed in terms of the polylogarithm, we briefly review this function.

For $s \in \mathbb{C}$, the polylogarithm function Li_s is defined by $Li_s(z) := \sum_{k=1}^{\infty} k^{-s} z^k$. It converges for $|z| < 1$. If s is a negative integer, say $s = -n$, then the polylogarithm function is a rational function in z and called the polypseudologarithm. The following formula is a simple consequence of Eq. (2.10c) in [1]:

$$\frac{d^m Li_{-n}(t)}{dt^m} = \frac{m! \sum_{j=0}^{n} (-1)^{n+j}(m+1)_j S(1+n, 1+j)(1-t)^{n-j}}{(1-t)^{n+m+1}}. \tag{3}$$

Here $S(\cdot,\cdot)$ denotes the Stirling number of the second kind.

Now we state our main result.

Theorem 2.1. *Under the conditions (1), (2), the Bergman kernel of $\Omega_{D_F,m}$ is given by*

$$K\big((z, \zeta), (z', \zeta')\big) = \frac{g(z, z')a(z, z')^{-sm}}{\pi^{m+2}} \frac{d^m}{dt^m} \sum_{n=0}^{2} c_n Li_{-n}(t)\bigg|_{t = a(z,z')^{-s}\langle \zeta, \zeta' \rangle},$$

where $c_0 = 2 + \gamma(1 - w)$, $c_1 = (c_0 + 1)s$ and $c_2 = s^2$.

Proof. The proof starts with the Forelli–Rudin construction, which is a series representation formula of the Hartogs domain (see [3] and [4]):

$$K\big((z, \zeta), (z', \zeta')\big) = \sum_{k=0}^{\infty} \frac{(k+1)_m}{\pi^m} K_{s(k+m)}(z, z')\langle \zeta, \zeta' \rangle^k, \tag{4}$$

where $(x)_m$ is the Pochhammer symbol. By (4) and Theorem 1.1, we know

$$K\big((z, \zeta), (z', \zeta')\big) = \frac{g(z, z')a(z, z')^{-sm}}{\pi^{m+2}} H_{s(k+m)}\big((z, \zeta), (z', \zeta')\big).$$

Here we put

$$H_c\big((z, \zeta), (z', \zeta')\big) = \sum_{k=0}^{\infty} (k+1)_m (c+1)\big[(c+2) + \gamma(1 - w)\big]\big[a(z, z')^{-c}\langle \zeta, \zeta' \rangle\big]^k, \quad \text{for } c > 0.$$

Then it is enough to show that

$$H_{s(k+m)}\big((z, \zeta), (z', \zeta')\big) = \frac{d^m}{dt^m} \sum_{n=0}^{2} c_n Li_{-n}(t)\bigg|_{t = a(z,z')^{-s}\langle \zeta, \zeta' \rangle}. \tag{5}$$

After a straightforward computation, we know that

$$H_{s(k+m)}\big((z,\zeta),(z',\zeta')\big)=\sum_{n=0}^{2}c_n\sum_{k=0}^{\infty}(k+1)_m(k+m)^n\big[a\big(z,z'\big)^{-s}\langle\zeta,\zeta'\rangle\big]^k. \tag{6}$$

From the definition of the polylogarithm it is easy to see that the m-th derivative of the polylogarithm function has the following series representation for $|z|<1$.

$$\frac{\mathrm{d}^m}{\mathrm{d}z^m}Li_{-n}(z)=\sum_{k=0}^{\infty}(k+1)_m(k+m)^n z^k. \tag{7}$$

Let $(z,\zeta),(z',\zeta')$ be elements of $\Omega_{D_F,m}$. Then the Cauchy–Schwarz inequality and (2) imply that

$$\big|\langle\zeta,\zeta'\rangle\big|^2\leqslant\|\zeta\|^2\big\|\zeta'\big\|^2<a(z,z)^s a\big(z',z'\big)^s<\big|a\big(z,z'\big)\big|^{2s}.$$

Consequently, we know that $|a(z,z')^{-s}\langle\zeta,\zeta'\rangle|<1$ for all $(z,\zeta),(z',\zeta')\in\Omega_{D_F,m}$. Combining (6) and (7), we obtain the formula (5). Thus we have completed the proof of this theorem. $\square$

Remark 1. In an analogous way, we can obtain explicit formulas of the Bergman kernels of the following Hartogs domains:

$$D_1=\big\{(z,\zeta)\in\mathbb{C}^n\times\mathbb{C}^m;\ \|\zeta\|^2<e^{-\|z\|^2}\big\},$$
$$D_2=\big\{(z,\zeta)\in D\times\mathbb{C}^m;\ \|\zeta\|^2<N(z,z)^s\big\}.$$

Here D is an irreducible bounded symmetric domain and N the generic norm of D. The domain D_1 (resp. D_2) is called the Fock–Bargmann–Hartogs domain (resp. the Cartan–Hartogs domain). It is known that the Bergman kernels of these domains are expressed in terms of the polylogarithm function. For further information, see [5] and [6].

We conclude this note with some examples which satisfy the conditions (1) and (2).

Example 1. $F(t)=(1-t)^p,\ p>0;\ B=1$. Then

$$D_F=\big\{z\in\mathbb{C}^2;\ |z_1|^2<1,\ |z_2|^2<\big(1-|z_1|^2\big)^p\big\},$$
$$\Omega_{D_F}=\big\{(z,\zeta)\in D_F\times\mathbb{C};\ |\zeta|^2<\big(\big(1-|z_1|^2\big)^p-|z_2|^2\big)^s\big\}.$$

A straightforward computation shows that the functions F,a satisfy the conditions (1) and (2) with $\gamma=\frac{1}{p}-1$. In this case $\{c_i\}_{i=0}^2$ is given by

$$c_0=\frac{1+p+(-1+p)w}{p},\qquad c_1=\frac{1+2p+(-1+p)w}{p}\cdot s,\qquad c_2=s^2.$$

Example 2. $F(t)=e^{-t}$, $B=+\infty$. Then

$$D_F=\big\{z\in\mathbb{C}^2;\ |z_2|^2<e^{-|z_1|^2}\big\},$$
$$\Omega_{D_F,m}=\big\{(z,\zeta)\in D_F\times\mathbb{C};\ |\zeta|^2<\big(e^{-|z_1|^2}-|z_2|^2\big)^s\big\}.$$

A straightforward computation shows that the functions F,a satisfy the conditions (1) and (2) with $\gamma=-1$. In this case $\{c_i\}_{i=0}^2$ is given by $c_0=1+w$, $c_1=s(2+w)$, $c_2=s^2$.

For the details of these examples, see [2, Section 4]. It would be interesting to find more examples.

References

[1] D. Cvijovic, Polypseudologarithms revisited, Physica A 389 (2010) 1594–1600.
[2] M. Engliš, Berezin quantization and reproducing kernels on complex domains, Trans. Amer. Math. Soc. 348 (1996) 411–479.
[3] E. Ligocka, On the Forelli–Rudin construction and weighted Bergman projection, Studia Math. 94 (1989) 257–272.
[4] G. Roos, Weighted Bergman kernels and virtual Bergman kernels, Sci. China Ser. A 48 (Suppl.) (2005) 387–399.
[5] A. Yamamori, A remark on the Bergman kernels of the Cartan–Hartogs domains, C. R. Acad. Sci. Paris, Ser. I 350 (2012) 157–160.
[6] A. Yamamori, The Bergman kernel of the Fock–Bargmann–Hartogs domain and polylogarithm function, Complex Var. Elliptic Equ., http://dx.doi.org/10.1080/17476933.2011.620098, in press.