C. R. Acad. Sci. Paris, Ser. I 350 (2012) 875–878

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Complex Analysis/Functional Analysis

Composition operators on Hilbert spaces of entire Dirichlet series

Opérateurs de composition sur les espaces de Hilbert de séries entières de Dirichlet

Hou Xiaolu, Hu Bingyang, Le Hai Khoi

Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University (NTU), 637371, Singapore

ARTICLE INFO

Article history:
Received 18 April 2012
Accepted after revision 9 October 2012
Available online 22 October 2012

Presented by the Editorial Board

ABSTRACT

In this Note, we introduce Hilbert spaces of entire Dirichlet series (with real frequencies) and consider composition operators on these spaces. We establish necessary and sufficient conditions for such series to have Ritt order zero, as well as a finite logarithmic order. Criteria for action, boundedness, compactness and compact difference of such operators are obtained.

RÉSUMÉ

Dans cette Note, nous introduisons des espaces de Hilbert de séries entières de Dirichlet (à fréquences réelles) et considérons des opérateurs de composition sur ces espaces. Nous donnons des conditions nécessaires et suffisantes sur de telles séries pour avoir un ordre de Ritt égal à zéro, ainsi qu'un ordre logarithmique fini. Nous obtenons des critères d'action, de continuité, de compacité pour de tels opérateurs ou leurs différences.

1. Introduction

1.1. Basic notation and definitions

Let $0 \leqslant (\lambda_n) \uparrow \infty$ be a given sequence of real positive numbers satisfying condition $L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n} < \infty$. As is well known, a Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \quad a_n, z \in \mathbb{C}, \tag{1}$$

represents an entire function in $\mathbb{C}$ if and only if $D = \limsup_{n\to\infty} \frac{\log|a_n|}{\lambda_n} = -\infty$.

In [2], for the class $\mathcal{H}^2(E) := \{f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}\colon\ D = -\infty\}$ of entire Dirichlet series, it was showed that $\mathcal{H}^2(E)$ is a normed space whose norm is defined by the inner product $\langle f, g\rangle := \sum_{n=1}^{\infty} a_n \bar{b}_n$, where $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ and $g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}$. Different properties of composition operators on this class, such as action, boundedness, were considered.

The important note is that the space $\mathcal{H}^2(E)$ is never complete with respect to the norm above. Then a natural question to ask is: what subspace of $\mathcal{H}^2(E)$ can be a Hilbert space?

E-mail addresses: HO0001LU@e.ntu.edu.sg (X. Hou), BHU2@e.ntu.edu.sg (B. Hu), lhkhoi@ntu.edu.sg (L.H. Khoi).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.10.012

We identify the space $\mathcal{H}^2(E)$ with the sequence space

$$E := \left\{(a_n)\colon \limsup \frac{\log |a_n|}{\lambda_n} = -\infty, \text{ or } \lim |a_n|^{1/\lambda_n} = 0\right\}.$$

To each sequence of real positive numbers $\beta = (\beta_n)$ we associate the following sequence space

$$X_\beta := \left\{a = (a_n)\colon \|a\| = \left(\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2\right)^{1/2} < \infty\right\},$$

which is a Hilbert (sequence) space with the inner product

$$\langle a, b\rangle = \sum_{n=1}^{\infty} a_n \bar{b}_n \beta_n^2, \quad \forall (a_n), (b_n) \in X_\beta \tag{2}$$

(see, e.g., [8]).

1.2. The main goal

Let $\mathcal{H}$ be a space of holomorphic functions on a set $G \subseteq \mathbb{C}$. If an analytic function φ maps G into itself, the *composition operator* C_φ is a linear operator defined by the rule

$$(C_\varphi f)(z) = f \circ \varphi(z), \quad z \in G, \ f \in \mathcal{H}.$$

Composition operators have been investigated on various spaces of holomorphic functions of one variable, as well as in higher dimensions. We refer the reader to the excellent monographs [1,7] for detailed information. In particular, an extensive study of composition operators was carried out on spaces of classical Dirichlet series.

The aim of this Note is to study composition operators on some Hilbert spaces of entire Dirichlet series whose coefficients are from X_β, under some natural assumptions on the sequence (β_n). It should be noted that we make use of Pólya's result on composition of entire functions [4], and therefore, the three notions of order for Dirichlet series (ordinary, Ritt and logarithmic orders) play an important role in our approach. It seems this topic has never been treated before.

2. Preliminaries and auxiliary results

First we prove the following elementary result (partially stated earlier in [3]), which is used very often in our discussion.

Lemma 2.1. *Let* $0 \leqslant (\lambda_n) \uparrow \infty$ *be given. The following conditions are equivalent*: (a) $\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = L < \infty$; (b) $\forall r > L$: $\sum_{n=1}^{\infty} e^{-r\lambda_n} < \infty$; (c) $\exists \rho > 0$: $\sum_{n=1}^{\infty} e^{-\rho\lambda_n} < \infty$; (d) $\limsup_{n\to\infty} \frac{\log n}{\lambda_n} \leqslant \rho$.

Put

$$\liminf_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = \beta_*, \qquad \limsup_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = \beta^*.$$

In [3] under the condition $L = 0$ (with complex frequencies $(\lambda_n) \subset \mathbb{C}$) there was given the characterization of the relationship between X_β and E in terms of β_* and β^*. By a similar method with an appropriate modification, we can prove that such a characterization is true in a general case $0 \leqslant L < +\infty$ for entire Dirichlet series (1).

Proposition 2.2. *There are three alternative possibilities*:

(1) *$X_\beta \subset E$, which is equivalent to $\beta_* = \infty$.*
(2) *$E \subset X_\beta$, which is equivalent to $\beta^* < \infty$.*
(3) *$X_\beta \setminus E \neq \emptyset$ and $E \setminus X_\beta \neq \emptyset$, which is equivalent to $\beta_* < \beta^* = \infty$, and therefore, these spaces never coincide.*

By Proposition 2.2, in the sequel the condition $\beta_* = \infty$ is always supposed to hold. That is, a sequence of positive real numbers (β_n) satisfies the condition

$$\liminf_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = \infty, \quad \text{or the same,} \quad \lim_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = \infty. \tag{E}$$

In this case we always have $X_\beta \subset E$, and by $\mathcal{H}^2(E, \beta)$ we denote the space of entire Dirichlet series with coefficients from X_β, that is

$$\mathcal{H}^2(E, \beta) := \left\{f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}\colon (a_n) \in X_\beta\right\}. \tag{3}$$

This is a Hilbert space with the inner product (2) above.

3. Ritt order and logarithmic orders

3.1. Different notions of orders for Dirichlet series

Let $f(z)$ be an entire function in $\mathbb{C}$. The ordinary order ρ of f is defined as the greatest lower bound of values of μ such that $\mathcal{M}_f(r) < e^{r^\mu}$ for all sufficiently large r, where $\mathcal{M}_f(r) = \max_{|z|=r}|f(z)|$, $r > 0$, is its maximum modulus (function). This order can be computed by $\rho = \limsup_{r\to\infty} \frac{\log\log\mathcal{M}_f(r)}{\log r}$.

For an entire Dirichlet series $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$, there exists the so-called Ritt order, which is determined, in particular, in terms of coefficients of the series as follows (see [6]): $\rho_R = \limsup_{\sigma\to-\infty} \frac{\log\log M(\sigma)}{-\sigma} = \limsup_{n\to\infty} \frac{\lambda_n \log\lambda_n}{\log\frac{1}{|a_n|}}$.

For entire Dirichlet series with Ritt order 0, Reddy defined, in particular, the following logarithmic order (see [5]): $\rho_c(\mathcal{L}) = \limsup_{n\to\infty} \frac{\log\lambda_n}{\log(\frac{1}{\lambda_n}\log\frac{1}{|a_n|})}$.

As noted in [2], for an entire Dirichlet series, if $\rho_R > 0$, then $\rho_c(\mathcal{L}) = \infty$, and hence there is not much information we can get from logarithmic orders. Perhaps that is why only entire Dirichlet series of the zero Ritt order are in consideration.

3.2. Ritt order for series from $\mathcal{H}^2(E,\beta)$

We are interested in the following question: when does each series from $\mathcal{H}^2(E,\beta)$ have Ritt order zero? The complete answer to this question is given by the following characterization.

Theorem 3.1. *Every function $f \in \mathcal{H}^2(E,\beta)$ has Ritt order zero if and only if the following condition (R) is satisfied:*

$$\liminf_{n\to\infty} \frac{\log\beta_n}{\lambda_n\log\lambda_n} = \infty. \tag{R}$$

Remark 3.2. Condition (R) is obviously stronger than condition (E). Moreover, the example $\beta_n = \lambda_n^{\lambda_n}$ $(n = 1, 2, \ldots)$, shows that the inverse implication is not true.

3.3. Logarithmic orders for series from $\mathcal{H}^2(E,\beta)$

Theorem 3.1 allows us to study the logarithmic orders of series from $\mathcal{H}^2(E,\beta)$. A question to ask is: under what condition all these series have a finite logarithmic order?

We have the following characterization.

Theorem 3.3. *Every function from $\mathcal{H}^2(E,\beta)$ has a finite logarithmic order if and only if the following condition (S) is satisfied*

$$\liminf_{n\to\infty} \frac{\log|\beta_n|}{\lambda_n^{1+\alpha}} = +\infty, \quad \textit{for some } \alpha > 0. \tag{S}$$

Remark 3.4. We can easily see that condition (S) is stronger than condition (R), and by the example $\beta_n = \lambda_n^{\lambda_n\log\lambda_n}$ $(n = 1, 2, \ldots)$, the inverse implication is not true.

4. Composition operators

4.1. Composition operators on the space $\mathcal{H}^2(E,\beta)$

An analytic mapping $\varphi : \mathbb{C} \to \mathbb{C}$ induces the composition operator C_φ acting, in particular, on the space $\mathcal{H}^2(E,\beta)$: $(C_\varphi(f))(z) = f\circ\varphi(z)$, $z \in \mathbb{C}$.

In [2] we considered the composition operators on the classes of entire Dirichlet series whose Ritt order is zero and logarithmic order is finite. The criteria for an action and boundedness are obtained.

The results obtained in the previous sections on Ritt order and logarithmic order of the series from $\mathcal{H}^2(E,\beta)$ allow us to consider composition operators in the later space. As the space $\mathcal{H}^2(E,\beta)$ is a Hilbert space, we can work also on other problems, such as compactness, compact difference, which we cannot have in [2].

By Theorems 3.1 and 3.3, in what follows, condition (S) is supposed to hold, that is

$$\liminf_{n\to\infty} \frac{\log|\beta_n|}{\lambda_n^{1+\alpha}} = +\infty, \quad \text{for some } \alpha > 0. \tag{S}$$

4.2. Criterion for the action

We note that series from $\mathcal{H}^2(E,\beta)$ have Ritt order zero and finite logarithmic orders. By [2, Theorem 3.2], the ordinary orders of these series are finite. This allows us to apply the Polya's result mentioned above, following the scheme in [2].

The first and also the most important issue to study is the action of C_φ on the space $\mathcal{H}^2(E,\beta)$. We consider a problem: for which φ does f in $\mathcal{H}^2(E,\beta)$ imply that $f\circ\varphi$ is also in $\mathcal{H}^2(E,\beta)$?

It turns out that the symbol φ can be only a translation. Namely, we have the following criterion.

Theorem 4.1. *Let $\varphi(z)$ be an entire function on $\mathbb{C}$. Let further, (β_n) satisfy condition (S). The composition operator $C_\varphi(f)=f\circ\varphi$ maps $\mathcal{H}^2(E,\beta)$ into itself if and only if $\varphi(z)$ is of the form*

$$\varphi(z)=z+b, \quad \textit{with } \operatorname{Re} b \geqslant 0. \tag{4}$$

The sufficiency is quite straightforward. The most difficult part is necessity. For proving this, we pass several steps, each of which in turn requires some auxiliary results. First, we show that if the composition operator $C_\varphi(f)=f\circ\varphi$ maps $\mathcal{H}^2(E,\beta)$ into itself, then $\varphi(z)=az+b$, where $a\geqslant 1$ satisfies the condition

$$\forall k\in\mathbb{N}\ \exists N(k) \quad \text{such that} \quad a=\frac{\lambda_{N(k)}}{\lambda_k}.$$

Next we prove that for the symbol $\varphi(z)$ in the formula above, the coefficient a must equal to 1, that is $\varphi(z)=z+b$, $b\in\mathbb{C}$. Finally, we show that $\varphi(z)=z+b$, the real part of b must be non-negative.

4.3. Criteria for boundedness, compactness and compact difference

4.3.1. Boundedness

From the proof of Theorem 4.1, it follows that when C_φ acts from $\mathcal{H}^2(E,\beta)$ into itself, this composition operator is automatically bounded on $\mathcal{H}^2(E,\beta)$. That is, we have the following result.

Theorem 4.2. *Let $\varphi(z)$ be an entire function on $\mathbb{C}$. Let further, (β_n) satisfy condition (S). The composition operator $C_\varphi(f)=f\circ\varphi$ is bounded on $\mathcal{H}^2(E,\beta)$ if and only if $\varphi(z)$ is of the form (4), that is*

$$\varphi(z)=z+b, \quad \textit{with } \operatorname{Re} b \geqslant 0.$$

4.3.2. Compactness

Theorem 4.3. *Let $\varphi(z)$ be an entire function on $\mathbb{C}$. Let further, (β_n) satisfy condition (S). The composition operator $C_\varphi(f)=f\circ\varphi$ is compact on $\mathcal{H}^2(E,\beta)$ if and only if*

$$\varphi(z)=z+b, \quad \textit{with } \operatorname{Re} b > 0.$$

4.3.3. Compact difference of composition operators

Finally, we study a compact difference of composition operators.

Theorem 4.4. *Suppose that $\varphi_1=z+b_1$, $\varphi_2=z+b_2$ with $\operatorname{Re} b_1\geqslant 0$, $\operatorname{Re} b_2\geqslant 0$ (that is composition operators C_{φ_1} and C_{φ_2} act from $\mathcal{H}^2(E,\beta)$ into itself). Then the difference $C_{\varphi_1}-C_{\varphi_2}$ is compact on $\mathcal{H}^2(E,\beta)$ if and only if either of the following conditions is satisfied*:

(1) *Both C_{φ_1} and C_{φ_2} are compact on $\mathcal{H}^2(E,\beta)$ (that is $\operatorname{Re} b_1>0$, $\operatorname{Re} b_2>0$).*
(2) *$\operatorname{Re} b_1=\operatorname{Re} b_2=0$ and $\lim_{n\to\infty}\cos[\lambda_n(\operatorname{Im} b_1-\operatorname{Im} b_2)]=1$.*

Acknowledgements

We would like to thank the referee(s) for useful remarks and comments that led to the improvement of this paper.

References

[1] C. Cowen, B. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, 1995.
[2] X. Hou, L.H. Khoi, Some properties of composition operators on entire Dirichlet series with real frequencies, C. R. Acad. Sci. Paris, Ser. I 350 (3–4) (2012) 149–152.
[3] L.H. Khoi, Hilbert spaces of holomorphic Dirichlet series and applications to convolution equations, J. Math. Anal. Appl. 206 (1) (1997) 10–24.
[4] G. Polya, On an integral function of an integral function, J. Lond. Math. Soc. 1 (1) (1926) 12.
[5] A.R. Reddy, On entire Dirichlet series of zero order, Tôhoku Math. J. (2) 18 (1966) 144–155.
[6] J.F. Ritt, On certain points in the theory of Dirichlet series, Amer. J. Math. 50 (1) (1928) 73–86.
[7] J.H. Shapiro, Compositions Operators and Classical Function Theory, Springer-Verlag, New York, 1993.
[8] A.L. Shields, Weighted shift operators and analytic function theory, in: Topics in Operator Theory, in: Math. Surveys, vol. 13, Amer. Math. Soc., Providence, RI, 1974, pp. 49–128.