## Algebra/Group Theory

## Jacquet modules of ladder representations

## Modules de Jacquet des représentations en échelle

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## A R T I C L E I N F O

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#### Abstract

We compute the Jacquet modules for a certain class of irreducible representations of the general linear group over a non-Archimedean local field. This class contains the Speh representations.


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## R É S U M É

On calcule les modules de Jacquet pour une certaine classe de représentations irréductibles du groupe linéaire général sur un corps local non-archimédien. Cette classe contient les représentations de Speh.
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## 1. Introduction

Let $F$ be a non-Archimedean local field with residue characteristic $p$ and consider the locally compact, totally disconnected group $G_{n}:=\mathrm{GL}_{n}(F)$. Let $P=M \ltimes N$ be the standard, block upper triangular, parabolic subgroup of type $\left(n_{1}, n_{2}, \ldots, n_{k}\right)$ with the standard Levi decomposition. Thus $M \simeq \prod_{i=1}^{k} G_{n_{i}}$. The normalized Jacquet functor $J_{P}$ is a functor from the category of smooth admissible complex representations of $G_{n}$ to those of $M$. It is defined as the space of coinvariants for the action of the unipotent group $N$ on $\pi$, twisted by a certain normalizing character. More precisely,

$$
J_{P}(\pi):=\pi_{N}\left[\delta_{P}^{-1 / 2}\right], \quad \text { where } \delta_{P}(m):=|\operatorname{det}(\operatorname{Ad}(m) \mid \operatorname{Lie}(N))|, \quad m \in M
$$

In general, it is a difficult problem to compute $J_{P}(\pi)$, or even its semisimplification, for an arbitrary irreducible $\pi$. In this note we will give an explicit formula for $J_{P}(\pi)$ for a certain class of irreducible representations, namely the ladder representations introduced in [6]. The case where $P$ is the minimal for which $J_{P}(\pi) \neq 0$ was considered in [6]. Here we will extend it to any $P$.

The class of ladder representations contains the class of Speh representations. The main result of [6] is to extend the determinantal formula of Tadić for Speh representations [8] (cf. also [2]) to ladder representations (see (2) below). Speh representations are important in the representation theory of the general linear group, because they form the building blocks for the unitary dual of $G_{n}$. More precisely, it was shown by Tadić that any irreducible unitary representation is isomorphic to the parabolic induction of Speh representations twisted by certain (explicit, but not necessarily unitary) characters [9].

[^0]In particular, this is the case for the local components of representations which occur in the discrete automorphic spectrum of $G_{n}$ over a global field.

It follows from our result that the Jacquet module of a ladder representation is semisimple, multiplicity free, and that its irreducible constituents are themselves tensor products of ladder representations. In contrast, the class of Speh representations is not stable under taking the Jacquet module. In other words, (non-Speh) ladder representations are encountered in the Jacquet module of Speh representations. Hence, ladder representations are important for global applications.

Our result has an application to Shimura varieties. In [5], the first named author computed the Hasse-Weil zeta function of the basic stratum of certain simple Shimura varieties at split primes of good reduction following the method of Langlands and Kottwitz [4]. Apart from the basic stratum, these varieties admit additional Newton strata (cf. [7]). In order to compute the zeta function of a given stratum $S$ one may proceed as in [5] provided that one knows the Jacquet modules of the representations occurring in the cohomology of $S$. These representations turn out to be (essentially) Speh representations, and hence the problem reduces to the one considered in this note. Details will be given elsewhere.

## 2. The main result

We first introduce some more notation. We write $\mathcal{R}=\bigoplus_{n \in \mathbb{Z} \geqslant 0} \mathcal{G} \mathcal{R}\left(G_{n}\right)$ where $\mathcal{G} \mathcal{R}\left(G_{n}\right)$ is the Grothendieck group of the category $\operatorname{Rep}\left(G_{n}\right)$ of smooth complex representations of $G_{n}$ of finite length. The group $\mathcal{R}$ has a structure of a graded ring (introduced by Zelevinsky in [10]) with multiplication given by

$$
\pi_{1} \times \pi_{2}:=\operatorname{Ind}_{P_{n_{1}, n_{2}}}^{G_{n_{1}+n_{2}}}\left(\pi_{1} \otimes \pi_{2}\right) \in \operatorname{Rep}\left(G_{n_{1}+n_{2}}\right),
$$

(normalized induction) for $\pi_{1} \in \operatorname{Rep}\left(G_{n_{1}}\right)$, $\pi_{2} \in \operatorname{Rep}\left(G_{n_{2}}\right), n_{1}, n_{2} \in \mathbb{Z} \geqslant 0$ where $P_{n_{1}, n_{2}}$ is the standard parabolic subgroup of $G_{n_{1}+n_{2}}$ of type ( $n_{1}, n_{2}$ ). The unit element of $\mathcal{R}$ is the one-dimensional representation of $G_{0}$.

Fix an integer $d>0$ and a cuspidal (not necessarily unitary) representation $\rho$ of $G_{d}$. For our purposes, a segment $[a, b]$ is a set of integers of the form $\{a, a+1, \ldots, b\}$ with $b \geqslant a$. For any segment $[a, b]$ the representation $\rho\left[|\operatorname{det} \cdot|^{a}\right] \times \cdots \times$ $\rho\left[\mid\right.$ det $\left.\left.\cdot\right|^{b}\right]$ admits a unique irreducible quotient $\delta([a, b])$, the so-called generalized Steinberg representation. A ladder is a finite sequence of segments $\left[a_{1}, b_{1}\right], \ldots,\left[a_{t}, b_{t}\right]$ such that $a_{1}>a_{2}>\cdots>a_{t}$ and $b_{1}>b_{2}>\cdots>b_{t}$. Given a ladder of segments, we may form the representation $\delta\left(\left[a_{1}, b_{1}\right]\right) \times \cdots \times \delta\left(\left[a_{t}, b_{t}\right]\right)$. This representation admits a unique irreducible quotient, $\operatorname{LQ}\left(\delta\left(\left[a_{1}, b_{1}\right]\right) \times \cdots \times \delta\left(\left[a_{t}, b_{t}\right]\right)\right)$, which is the Langlands quotient in the case at hand. The representations which arise in this manner are by definition the ladder representations. The subclass of Speh representations (up to twists) is obtained by taking $a_{i+1}=a_{i}-1$ and $b_{i+1}=b_{i}-1$ for all $i=1, \ldots, t-1$.

The ring $\mathcal{R}$ is actually a bi-algebra (and in fact has an additional structure of a Hopf-algebra) with respect to the comultiplication $\Delta: \mathcal{R} \rightarrow \mathcal{R} \otimes \mathcal{R}$ defined by $\pi \mapsto \sum_{i=0}^{n} J_{P_{i, n-i}}(\pi)$, $\pi \in \operatorname{Rep}\left(G_{n}\right)$. In particular we have

$$
\begin{equation*}
\Delta(\delta([a, b]))=\sum_{c} \delta([c+1, b]) \otimes \delta([a, c]) \tag{1}
\end{equation*}
$$

where we have used the convention that $\delta([a, b])=0$ if $b<a-1$ and $\delta([a, a-1])=1 \in \mathcal{R}$.
Theorem 2.1. Suppose that $a_{1}>\cdots>a_{t}$ and $b_{1}>\cdots>b_{t}$. Then

$$
\begin{aligned}
\Delta & \left(\operatorname{LQ}\left(\delta\left(\left[a_{1}, b_{1}\right]\right), \ldots, \delta\left(\left[a_{t}, b_{t}\right]\right)\right)\right) \\
& =\sum_{c_{1}>\cdots>c_{t} \in \mathbb{Z}} \operatorname{LQ}\left(\delta\left(\left[c_{1}+1, b_{1}\right]\right), \ldots, \delta\left(\left[c_{t}+1, b_{t}\right]\right)\right) \otimes \operatorname{LQ}\left(\delta\left(\left[a_{1}, c_{1}\right]\right), \ldots, \delta\left(\left[a_{t}, c_{t}\right]\right)\right) .
\end{aligned}
$$

Remark 1. Note the similarity between this formula and the formula

$$
\Delta\left(\delta\left(\left[a_{1}, b_{1}\right]\right) \times \cdots \times \delta\left(\left[a_{t}, b_{t}\right]\right)\right)=\sum_{c_{1}, \ldots, c_{t} \in \mathbb{Z}} \delta\left(\left[c_{1}+1, b_{1}\right]\right) \times \cdots \times \delta\left(\left[c_{t}+1, b_{t}\right]\right) \otimes \delta\left(\left[a_{1}, c_{1}\right]\right) \times \cdots \times \delta\left(\left[a_{t}, c_{t}\right]\right)
$$

Let us now prove Theorem 2.1. By the determinantal formula of Tadić [6] we have

$$
\begin{equation*}
\operatorname{LQ}\left(\delta\left(\left[a_{1}, b_{1}\right]\right), \ldots, \delta\left(\left[a_{t}, b_{t}\right]\right)\right)=\operatorname{det}\left(\delta\left(\left[a_{i}, b_{j}\right]\right)\right)_{i, j=1, \ldots, t} \tag{2}
\end{equation*}
$$

Since $\Delta$ is an algebra homomorphism we get

$$
\Delta\left(\operatorname{LQ}\left(\delta\left(\left[a_{1}, b_{1}\right]\right), \ldots, \delta\left(\left[a_{t}, b_{t}\right]\right)\right)\right)=\operatorname{det}\left(\Delta\left(\delta\left(\left[a_{i}, b_{j}\right]\right)\right)\right)_{i, j=1, \ldots, t} .
$$

By (1) and using the multi-linearity of the determinant, it is further equal to

$$
\sum_{c_{1}, \ldots, c_{t} \in \mathbb{Z}} \operatorname{det}\left(\delta\left(\left[c_{j}+1, b_{j}\right]\right) \otimes \delta\left(\left[a_{i}, c_{j}\right]\right)\right)=\sum_{c_{1}, \ldots, c_{t} \in \mathbb{Z}}\left(\prod_{j=1}^{t} \delta\left(\left[c_{j}+1, b_{j}\right]\right)\right) \otimes \operatorname{det}\left(\delta\left(\left[a_{i}, c_{j}\right]\right)\right)
$$

Write $S_{t}$ for the symmetric group on the set $\{1,2, \ldots, t\}$. Observe that if $c_{j}=c_{k}$ for some $j \neq k$ then $\operatorname{det}\left(\delta\left(\left[a_{i}, c_{j}\right]\right)\right)=0$ since two columns in the matrix are identical. Therefore, only distinct $c_{1}, \ldots, c_{t}$ contribute to the right-hand side of the above equation, and we can write the sum as

$$
\begin{aligned}
& \sum_{c_{1}>\cdots>c_{t} \in \mathbb{Z}} \sum_{s \in S_{t}}\left(\prod_{j=1}^{t} \delta\left(\left[c_{s(j)}+1, b_{j}\right]\right)\right) \otimes \operatorname{det}\left(\delta\left(\left[a_{i}, c_{s(j)}\right]\right)\right) \\
= & \sum_{c_{1}>\cdots>c_{t} \in \mathbb{Z}} \sum_{s \in S_{t}} \operatorname{sgn} s\left(\prod_{j=1}^{t} \delta\left(\left[c_{s(j)}+1, b_{j}\right]\right)\right) \otimes \operatorname{det}\left(\delta\left(\left[a_{i}, c_{j}\right]\right)\right) \\
= & \sum_{c_{1}>\cdots>c_{t} \in \mathbb{Z}} \operatorname{det}\left(\delta\left(\left[c_{i}+1, b_{j}\right]\right)\right) \otimes \operatorname{det}\left(\delta\left(\left[a_{i}, c_{j}\right]\right)\right) .
\end{aligned}
$$

Applying (2) once more, we obtain Theorem 2.1.
Corollary 2.2. Suppose that $a_{1}>\cdots>a_{t}$ and $b_{1}>\cdots>b_{t}$. Then the Jacquet module of $\operatorname{LQ}\left(\delta\left(\left[a_{1}, b_{1}\right]\right), \ldots, \delta\left(\left[a_{t}, b_{t}\right]\right)\right)$ with respect to the parabolic subgroup of type $\left(n_{1}, \ldots, n_{k}\right)$ is

$$
\begin{equation*}
\bigoplus_{f} \operatorname{LQ}\left(f^{-1}(1)\right) \otimes \cdots \otimes \operatorname{LQ}\left(f^{-1}(k)\right) \tag{3}
\end{equation*}
$$

where the sum is over all $k$-colorings $f: \bigcup_{i=1}^{t}\left(\left[a_{i}, b_{i}\right] \times\{i\}\right) \rightarrow\{1, \ldots, k\}$ such that
(i) $j \mapsto f(j, i)$ is (weakly) monotone decreasing for all $i=1, \ldots, t$,
(ii) $n_{l}=d \cdot\left|f^{-1}(l)\right|$ for all $l=1, \ldots, k$,
(iii) for any $l=1, \ldots, k$ and $i=1, \ldots, t$, let $m_{i, l}=\min \left\{j \in\left[a_{i}, b_{i}+1\right]: f(j, i) \leqslant l\right\}\left(\right.$ with $\left.f\left(b_{i}+1, i\right)=-\infty\right)$ and $n_{i, l}=\max \{j \in$ $\left.\left[a_{i}-1, b_{i}\right]: f(j, i) \geqslant l\right\}\left(\right.$ with $\left.f\left(a_{i}-1, i\right)=\infty\right)$. Then $m_{i, l}>m_{i+1, l}$ and $n_{i, l}>n_{i+1, l}$ for all $i=1, \ldots, t-1, l=1, \ldots, k$.

## See Fig. 1.

The corollary extends the result of [6] (i.e., the case $n_{1}=\cdots=n_{t}=d$ ). Up to semisimplification, the corollary follows from Theorem 2.1 by induction on $k$. To show that the Jacquet module is semisimple it suffices to note that the summands in (3) have distinct supercuspidal supports. This follows from the fact that given $b_{1}>\cdots>b_{t}$ and a multiset $A$ of integers, there is at most one sequence $a_{1}>\cdots>a_{t}$ such that $a_{i} \leqslant b_{i}+1$ for all $i$ and $A=\bigcup\left[a_{i}, b_{i}\right]$. We apply this inductively on $l$ to show that $m_{i, l}$ and $n_{i, l}, i=1, \ldots, t$ are determined by the supercuspidal support.

Fig. 1. An example of a 4 -coloring of 3 segments satisfying the conditions of Corollary 2.2.
Fig. 1. Un exemple d'une 4-coloration de 3 segments satisfaisant les conditions du Corollaire 2.2.

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