Partial Differential Equations/Mathematical Problems in Mechanics

# A periodic unfolding operator on certain compact Riemannian manifolds 

# Un opérateur d'éclatement périodique pour quelques variétés riemanniennes compactes 

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#### Abstract

In this note, we present a generalisation of the method of periodic unfolding, which can be applied to structures defined on certain compact Riemannian manifolds. While many results known from unfolding in domains of $\mathbb{R}^{n}$ can be recovered, for the unfolding of gradients a transport operator has to be defined.


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On propose une généralisation de la méthode d'éclatement périodique qui peut être appliquée aux structures définies sur quelques variétés riemanniennes compactes. Tandis que la pluspart des résultats connus de l' éclatement périodique dans un domain en $\mathbb{R}^{n}$ est également valide, on a besoin d'un opérateur de transport pour l'éclatement des gradients.
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## 1. Introduction

The notion of periodic unfolding was introduced by Cioranescu, Damlamian, and Griso [3] in 2002 and has become a standard tool in the theory of homogenisation (see also [5] for an introduction).

In this note, we present a generalisation of the notion of periodic unfolding which is applicable to certain classes of compact Riemannian manifolds. This way, we can obtain homogenisation results for partial differential equations posed for example on nonflat surfaces. Obviously, there is no natural notion of periodicity available in this case. Here, we rely on a representation in local coordinates. Full details of the proofs together with several applications can be found in [6] and will be presented in forthcoming publications.

## 2. Notation and basic definitions

In the sequel, we will consider a smooth compact manifold $M \subset \mathbb{R}^{m}$ (with or without boundary) of dimension $n$, with $m, n \in \mathbb{N}$. Denote an atlas by $\mathcal{A}=\left\{\left(U_{\alpha}, \phi_{\alpha}\right) ; \alpha \in I\right\}$ with charts $\phi_{\alpha}: U_{\alpha} \rightarrow \mathbb{R}^{n}$ for some finite index set $I$. Note that we are not considering equivalence classes of atlases, but we stick to a fixed one. The tangent space $T_{x} M$ in $x \in M$ is given by the span of the vectors $v$, such that $v=\frac{\mathrm{d} \gamma}{\mathrm{d} t}(0)$ for a differentiable curve $\gamma:(-\delta, \delta) \rightarrow M$ with $\gamma(0)=x$ and $\delta>0$. Using the curves $t \mapsto \phi^{-1}\left(\phi(x)+t e_{i}\right)$, they give rise to the so-called local basis vectors $\frac{\partial}{\partial x^{i}}(x)$ for $i=1, \ldots, n, \phi \in \mathcal{A}$.

[^0]On $M$, let there be given a smooth Riemannian metric $g_{M} \in \Gamma\left(T^{*} M \otimes T^{*} M\right)$, i.e. $g_{M}(x)$ defines a scalar product on the tangent space $T_{x} M, x \in M$. Let $g_{M}$ be given in local coordinates by the expression $g_{M}=\sum_{i, j=1}^{n} g_{i j} \mathrm{~d} x^{i} \otimes \mathrm{~d} x^{j}$. Here $g_{i j}(x)=g_{M}(x)\left(\frac{\partial}{\partial x^{i}}(x), \frac{\partial}{\partial x^{j}}(x)\right)$, and $\left(\mathrm{d} x^{i}(x)\right)_{i=1, \ldots, n}$ constitutes the dual basis to $\left(\frac{\partial}{\partial x^{i}}(x)\right)_{i=1, \ldots, n}$ at $x \in M$. Defining the matrix $G=\left[g_{i j}\right]_{i, j=1, \ldots, n}$, we denote the volume form on $M$ by dvol ${ }_{M}=\sqrt{|\operatorname{det} G|} \mathrm{d} x$ (where $\mathrm{d} x=\mathrm{d} x^{1} \wedge \cdots \wedge \mathrm{~d} x^{n}$ ).

For a chart $\phi$, we denote by $\phi_{*}$ the pushforward operator, and by $\phi^{*}$ the pullback operator induced by $\phi$. For scalar functions $f: U \rightarrow \mathbb{R}$ with chart $\phi: U \rightarrow \mathbb{R}^{n}$, we have $\phi_{*} f(z)=f\left(\phi^{-1}(z)\right)$ and $\phi^{*}=\left(\phi^{-1}\right)_{*}$ for $z \in \phi(U)$. Further details of these notions can be found in [2], for example. For $M$ being a domain in $\mathbb{R}^{n}$, one can think of $\phi=\operatorname{Id}, \frac{\partial}{\partial x^{i}} \equiv e_{i}$ and $g_{M}$ as the usual Euclidean scalar product. Lebesgue and Sobolev spaces for scalar functions on $M$ are denoted by $L^{p}(M)$ and $W^{s, p}(M)$, respectively, where $p \in[1, \infty]$ denotes the (generalised) order of integrability and $s \geqslant 0$ the order of differentiability. Similarly, we denote the Lebesgue space of vector fields by $L^{p} T M$. The subscript \# indicates periodicity.

Finally, denote by $Y:=[0,1)^{n}$ the reference cell in $\mathbb{R}^{n}$, endowed with the topology of the torus. We use the following notation from the usual unfolding theory: For $x \in \mathbb{R}^{n}$, the representation $x=\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon\left\{\frac{x}{\varepsilon}\right\}$ holds, where $[z]=\left(b_{1}, \ldots, b_{n}\right) \in \mathbb{Z}^{n}$ is the unique vector such that $\{z\}:=z-[z] \in Y$. For a function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$, the unfolded function $\mathcal{T}^{\varepsilon}: \mathbb{R}^{n} \times Y \rightarrow \mathbb{R}$ is defined by $\mathcal{T}^{\varepsilon}(f)(x, y)=f\left(\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y\right)$. We consider a fixed sequence $\varepsilon_{n} \rightarrow 0$ with $\varepsilon_{n}>0, n \in \mathbb{N}$. As usual in the theory of homogenisation, we denote this sequence and its elements by $\varepsilon$.

## 3. Unfolding operators on $M$

The method of periodic unfolding is usually applied to (flat) domains which have an $\varepsilon$-periodic structure. For our generalisation to manifolds, we also have a periodic structure in mind. We say that an object is $\varepsilon_{\mathcal{A}}$-periodic, if it is $\varepsilon Y$-periodic in $\mathbb{R}^{n}$ after transformation with a chart $\phi$ from the designated atlas $\mathcal{A}$. For example, if we take a smooth $\varepsilon Y$-periodic function $f: Y \rightarrow \mathbb{R}$ and a $\phi_{\alpha} \in \mathcal{A}$, then $\phi_{\alpha}^{*} f=f \circ \phi_{\alpha}$ is $\varepsilon_{\mathcal{A}}$-periodic on $U_{\alpha}$. One can also think of $M$ itself being $\varepsilon_{\mathcal{A}}$-periodic, if we image $M$ to represent a material body whose properties (for example heat conductivity etc.) vary in an $\varepsilon_{\mathcal{A}}$-periodic way. Similar ideas can be found in the work of Neuss, Neuss-Radu, and Mikelić [7]. We need the following compatibility condition:

Definition 3.1 (UC-atlas). The atlas $\mathcal{A}$ is said to be compatible with unfolding (UC) if for all $\alpha, \beta \in I$ with $U_{\alpha} \cap U_{\beta} \neq \emptyset$ and for all $\varepsilon$ there exists a $k(\varepsilon) \in \mathbb{Z}^{n}$, possibly depending on $\varepsilon$, such that $\phi_{\alpha}=\phi_{\beta}+\varepsilon \sum_{i=1}^{n} k_{i}(\varepsilon) e_{i}$ in $U_{\alpha} \cap U_{\beta}$, where $e_{i}$ denotes the $i$-th unit vector in $\mathbb{R}^{n}$.

For example, one can endow a circle in $\mathbb{R}^{2}$ or a spherical zone in $\mathbb{R}^{3}$ with a UC-atlas $\mathcal{A}$. The following two lemmas follow by the definition of $\{\cdot\}$ and the chain rule:

Lemma 3.2. Let $\phi_{\alpha}$ and $\phi_{\beta}$ be two charts of a UC-atlas $\mathcal{A}$ with $U_{\alpha} \cap U_{\beta} \neq \emptyset$. For all admissible $\varepsilon$ and $x \in U_{\alpha} \cap U_{\beta}$ it holds $\left\{\frac{\phi_{\alpha}(x)}{\varepsilon}\right\}=$ $\left\{\frac{\phi_{\beta}(x)}{\varepsilon}\right\}$.

Lemma 3.3. Let $\phi_{\alpha}=\left(x^{1}, \ldots, x^{n}\right)$ and $\phi_{\beta}=\left(\tilde{x}^{1}, \ldots, \tilde{x}^{n}\right)$ be two charts of a UC-atlas $\mathcal{A}$ with $U_{\alpha} \cap U_{\beta} \neq \emptyset$. For the local basis vectors, the identity $\frac{\partial}{\partial x^{i}}=\frac{\partial}{\partial \tilde{x}^{i}}, i=1, \ldots, n$, holds across different charts in $U_{\alpha} \cap U_{\beta}$.

We now define local unfolding operators for functions, vector fields and forms. The main idea is to apply the "usual" unfolding operator $\mathcal{T}^{\varepsilon}$ after transformation to $\mathbb{R}^{n}$, followed by a pullback to the manifold $M$ of the object thus defined.

Definition 3.4. Choose a chart $\phi \in \mathcal{A}$ with corresponding domain $U \subset M$.
(i) For a function $f: U \rightarrow \mathbb{R}$ we define $\mathcal{T}_{\phi}^{\varepsilon}(f)=(\phi \times \mathrm{Id})^{*} \mathcal{T}^{\varepsilon}\left(\phi_{*} f\right): U \times Y \rightarrow \mathbb{R}$.
(ii) For a vector field $F \in \Gamma(T U)$ define analogously the vector field on $Y \mathcal{T}_{\phi}^{\varepsilon}(F)=(\phi \times \mathrm{Id})^{*} \mathcal{T}^{\varepsilon}\left(\phi_{*} F\right) \in \Gamma(T Y)^{U}$.
(iii) For a $k$-form $\eta \in \Omega^{k}(U)$ on $U$ with $\eta=\sum_{(j)} a^{(j)} \mathrm{d} x^{(j)}$ set $\mathcal{T}_{\phi}^{\varepsilon}(\eta)=\sum_{(j)} \mathcal{T}_{\phi}^{\varepsilon}\left(a^{(j)}\right) \mathrm{d} y^{(j)} \in \Omega^{k}(Y)^{U}$ to be the unfolded $k$-form on $Y$, where the forms $\mathrm{d} y^{(j)}$ stem from the trivial chart Id for $Y$.

One can show that for a vector field $F=\sum_{i=1}^{n} F^{i} \frac{\partial}{\partial x^{i}}$, the unfolded vector field in local coordinates is given by $\mathcal{T}_{\phi}^{\varepsilon}(F)=$ $\sum_{i=1}^{n} \mathcal{T}_{\phi}^{\varepsilon}\left(F^{i}\right) \frac{\partial}{\partial y^{i}}$, resembling the definition of $\mathcal{T}_{\phi}^{\varepsilon}(\eta)$. Obviously, we also have $\mathcal{T}_{\text {Id }}^{\varepsilon}=\mathcal{T}^{\varepsilon}$.

Definition 3.5. Let $\left\{\pi_{\alpha}, \alpha \in I\right\}$ be a partition of unity subordinate to the covering $\left\{U_{\alpha}, \alpha \in I\right\}$. The global unfolding operator $\mathcal{T}_{\mathcal{A}}^{\varepsilon}$ with respect to a UC-atlas $\mathcal{A}$ is defined as $\mathcal{T}_{\mathcal{A}}^{\mathcal{\varepsilon}}(\cdot)=\sum_{\alpha \in I} \mathcal{T}_{\phi_{\alpha}}^{\varepsilon}\left(\pi_{\alpha}\right) \mathcal{T}_{\phi_{\alpha}}^{\varepsilon}\left(\left.\cdot\right|_{U_{\alpha}}\right)$.

Note that due to the UC-criterion, $\mathcal{T}_{\mathcal{A}}^{\varepsilon}$ is well defined. Equivalent definitions can be found in [6].

### 3.1. Operator properties and integral identities

The operator $\mathcal{T}_{\mathcal{A}}^{\varepsilon}$ has similar properties as the usual unfolding operator $\mathcal{T}^{\varepsilon}$, which we collect in this part of the note:
For two scalar functions $f, g: M \rightarrow \mathbb{R}$ the identities $\mathcal{T}_{\mathcal{A}}^{\varepsilon}(f+g)=\mathcal{T}_{\mathcal{A}}^{\varepsilon}(f)+\mathcal{T}_{\mathcal{A}}^{\varepsilon}(g)$ and $\mathcal{T}_{\mathcal{A}}^{\varepsilon}(f \cdot g)=\mathcal{T}_{\mathcal{A}}^{\varepsilon}(f) \cdot \mathcal{T}_{\mathcal{A}}^{\varepsilon}(g)$ hold. A simple calculation shows that the unfolding of an integral expression yields a remainder $r \in \mathbb{R}$ for each fixed $\varepsilon$. Thus, similarly to the unfolding criterion for integrals in [4] we have the following:

Definition 3.6 (UCM-criterion). A sequence $\left\{f^{\varepsilon}\right\}$ in $L^{1}(M)$ is said to fulfill the unfolding criterion on manifolds (UCM) if there exists a function $r: \mathbb{R}^{+} \rightarrow \mathbb{R}$ such that $r(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and

$$
\int_{M} f^{\varepsilon} \operatorname{dvol}_{M}=\frac{1}{|Y|} \int_{M \times Y} \mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(f^{\varepsilon}\right) \mathcal{T}_{\mathcal{A}}^{\varepsilon}(\sqrt{|G|}) \mathrm{d} y \mathrm{~d} x+r(\varepsilon)
$$

We write in this case $\int_{M} f^{\varepsilon} \operatorname{dvol}_{M} \simeq \frac{1}{|Y|} \int_{M \times Y} \mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(f^{\varepsilon}\right) \mathcal{T}_{\mathcal{A}}^{\varepsilon}(\sqrt{|G|}) \mathrm{d} y \mathrm{~d} x$.
Note that the last part of the right hand side corresponds to the "unfolded" volume form on $M$. Together with the next proposition, one can show that any fixed function $f \in L^{1}(M)$ as well as any sequence $\left\{f^{\varepsilon}\right\} \subset L^{p}(M), 1 \leqslant p \leqslant \infty$, which is bounded independently of $\varepsilon$ fulfills the UCM-criterion.

Theorem 3.7. Fix $1 \leqslant p<\infty$. For all $\delta>0$ there exists an $\varepsilon_{0}(\delta)>0$ such that for all $\varepsilon<\varepsilon_{0}(\delta)$ the scalar unfolding operators $\mathcal{T}_{\mathcal{A}}^{\varepsilon}: L^{p}(M) \rightarrow L^{p}(M \times Y)$ are linear and continuous with operator norm bounded by $[(1+\operatorname{card}(I) \delta)|Y|]^{\frac{1}{p}}$.

Here the volume measure on $M \times Y$ is the measure induced by the form $\mathrm{dvol}_{M} \mathrm{~d} y$. Using the results from Section 3.2, one can show that the same result holds true for the unfolding operators acting on vector fields $\mathcal{T}_{\mathcal{A}}^{\varepsilon}: L^{p} T M \rightarrow L^{p}\left(M ; L^{p} T Y\right)$.

The following results are well-known for the standard unfolding operator $\mathcal{T}^{\varepsilon}$ :
Lemma 3.8. Fix $p \in[1, \infty)$ and let $\left\{w^{\varepsilon}\right\} \subset L^{p}(M)$ be a sequence.

- If $w^{\varepsilon} \rightarrow w$ strongly in $L^{p}(M)$ with some $w \in L^{p}(M)$, then $\mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(w^{\varepsilon}\right) \rightarrow w$ strongly in $L^{p}(M \times Y)$. The same result also applies to the case $p=\infty$.
- Assume that $\mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(w^{\varepsilon}\right) \rightharpoonup \hat{w}$ weakly in $L^{p}(M \times Y)$ with some $\hat{w} \in L^{p}(M \times Y)$. Then $w^{\varepsilon}$ converges weakly in $L^{p}(M)$ to $w$, where $w=\int_{Y} \hat{w} \mathrm{~d} y$.

The next result corresponds to the usual unfolding of gradients. However, as on manifolds the "natural" differential operator is the exterior derivative, we start by considering forms:

Proposition 3.9. Let $\eta \in \Omega^{r}(M)$ be an $r$-form, $r \in \mathbb{N}_{0}$. Then for the exterior derivative d it holds $\varepsilon \mathcal{T}_{\mathcal{A}}^{\varepsilon}(\mathrm{d} \eta)=\mathrm{d}_{y} \mathcal{T}_{\mathcal{A}}^{\varepsilon}(\eta)$.
Proof. For $\eta \in \Omega^{0}(M)$ one has $\mathrm{d} \eta=\sum_{i=1}^{n} \frac{\partial \eta}{\partial x^{i}} \mathrm{~d} x^{i}$. Since locally $\varepsilon \mathcal{T}_{\phi}^{\varepsilon}\left(\frac{\partial \eta}{\partial x^{i}}\right)=\frac{\partial \mathcal{T}_{\phi}^{\varepsilon}(\eta)}{\partial y^{i}}$, the result follows for 0 -forms. For the general case, use the representation from Definition 3.4(iii).

### 3.2. Unfolding of gradients in the Hilbert space case

In order to be able to prove unfolding results for gradients, we need to define two parameter-dependent Riemannian metrics: For fixed $x \in M$ and $\varepsilon$ define Riemannian metrics on the reference cell $Y$ via

$$
g_{Y}^{(x, \varepsilon)}:=\mathcal{T}_{\mathcal{A}}^{\mathcal{E}}\left(g_{M}\right)(x, \cdot):=\sum_{i, j=1}^{n} \mathcal{T}_{\mathcal{A}}^{\mathcal{E}}\left(g_{i j}\right)(x, \cdot) \mathrm{d} y^{i} \otimes \mathrm{~d} y^{j} \quad \text { as well as } \quad g_{Y}^{(x)}:=\sum_{i, j=1}^{n} g_{i j}(x) \mathrm{d} y^{i} \otimes \mathrm{~d} y^{j}
$$

These metrics stem from the unfolding of the metric coefficients $g_{i j}, i, j=1, \ldots, n$.
Gradients are defined as dual operators of the exterior derivative with respect to a Riemannian metric. Thus, we denote the gradient on $M$ with respect to $g_{M}$ by $\nabla_{M}$ as well as the gradient on $Y$ with respect to $g_{Y}^{(x, \varepsilon)}$ (and $g_{Y}^{(x)}$ ) by $\nabla_{Y}^{(x, \varepsilon)}$ (and $\nabla_{Y}^{(x)}$, respectively). By considering the expressions in local coordinates, one shows the following results:

Lemma 3.10. Let $F$ and $G$ be two vector fields. It holds $\mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(g_{M}(F, G)\right)(x, y)=g_{Y}^{(x, \varepsilon)}\left(\mathcal{T}_{\mathcal{A}}^{\varepsilon}(F)(x, y), \mathcal{T}_{\mathcal{A}}^{\varepsilon}(G)(x, y)\right)$.
Proposition 3.11. Let $f: M \rightarrow \mathbb{R}$ be a differentiable function, and let $F \in \Gamma(T M)$ be a differentiable vector field. Then the identities $\varepsilon \mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(\nabla_{M} f\right)(x, y)=\nabla_{Y}^{(x, \varepsilon)} \mathcal{T}_{\mathcal{A}}^{\varepsilon}(f)(x, y)$ and $\varepsilon \mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(\operatorname{div}_{M} F\right)(x, y)=\operatorname{div}_{Y}^{(x, \varepsilon)} \mathcal{T}_{\mathcal{A}}^{\varepsilon}(F)(x, y)$ hold, where $\operatorname{div}_{M}$ denotes the divergenceoperator on $M$ with respect to $g_{M}$ (similarly for $g_{Y}^{(x, \varepsilon)}$ ).

When dealing with unfolding results for gradients, one distinguishes the two cases of "weak" estimates $\left\|w^{\varepsilon}\right\|+$ $\varepsilon\left\|\nabla w^{\varepsilon}\right\| \leqslant C$ and the case of "strong" estimates $\left\|w^{\varepsilon}\right\|+\left\|\nabla w^{\varepsilon}\right\| \leqslant C$. For the case of the weak estimates, one obtains the following result:

Theorem 3.12. Let $\left\{w^{\varepsilon}\right\} \subset W^{1,2}(M)$ be a sequence such that $\left\|w^{\varepsilon}\right\|_{L^{2}(M)} \leqslant C$ and $\varepsilon\left\|\nabla_{M} w^{\varepsilon}\right\|_{L^{2} T M} \leqslant C$ with a constant $C$ independent of $\varepsilon$. Then there exists $a w \in L^{2}\left(M ; W_{\#}^{1,2}(Y)\right)$ such that along a subsequence (still denoted by $\varepsilon$ )

$$
\mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(w^{\varepsilon}\right) \rightharpoonup w \quad \text { weakly in } L^{2}(M \times Y), \quad \varepsilon \mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(\nabla_{M} w^{\varepsilon}\right) \rightharpoonup \nabla_{Y}^{(x)} w \quad \text { weakly in } L^{2}\left(M ; L^{2} T Y\right)
$$

where by abuse of notation we use $\nabla_{Y}^{(x)} w$ to denote the function $(x, y) \mapsto \nabla_{Y}^{(x)} w(x, y)$.
Proof. Due to the boundedness and Proposition 3.11, there exist limits $w \in L^{2}(M \times Y)$ and $\xi \in L^{2}\left(M ; L^{2} T Y\right)$ such that along a subsequence $\mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(w^{\varepsilon}\right) \rightharpoonup w$ in $L^{2}(M \times Y), \nabla_{Y}^{(x, \varepsilon)} \mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(w^{\varepsilon}\right) \rightharpoonup \xi$ in $L^{2}\left(M ; L^{2} T Y\right)$. To obtain that $\xi=\nabla_{Y}^{(x)} w$, one shows that $\xi$ is orthogonal to functions $\psi \in L^{2}\left(M ; W_{\#}^{1,2} T Y\right)$ with $\operatorname{div}_{Y}^{(x)}(\psi)=0$. Using Hodge-theory (see e.g. [1]), one sees that $\xi$ can be represented as $\xi=\nabla_{Y}^{(x)} \zeta$ for some $\zeta \in L^{2}\left(M ; W_{\#}^{1,2} T Y\right)$. In the same way, one then obtains that $w-\zeta$ is orthogonal to $L^{2}$-function in $M \times Y$ with mean value 0 , i.e. $w=\zeta+K$ with some constant $K \in \mathbb{R}$. This finally yields $\nabla_{Y}^{(x)} w=\nabla_{Y}^{(x)} \zeta=\xi$.

For the case of stronger estimates, one expects a result like $\mathcal{T}^{\varepsilon}\left(\nabla w^{\varepsilon}\right)(x, y) \rightharpoonup \nabla w(x)+\nabla_{y} \hat{w}(x, y)$. This expression, however, cannot be directly transferred to the manifold case: A term of the form $\nabla_{M} w(x)+\nabla_{Y}^{(x)} \hat{w}(x, y)$ is a sum of a vector field on $M$ and a vector field on $Y$ - i.e. two quantities which are not a-priori related! Therefore, a transport operator appears, mapping vector fields on $M$ to vector fields on $Y$ :

Definition 3.13. For a parameter-dependent vector field $F \in \Gamma(T M)^{Y}$ we define a transport operator $(\cdot)_{Y}$ with $(\cdot)_{Y}$ : $\Gamma(T M)^{Y} \rightarrow \Gamma(T Y)^{M}, F \mapsto F_{Y}$, where for $F=\sum_{i} F^{i} \frac{\partial}{\partial x^{i}}$ the vector field $F_{Y}$ on $Y$ is defined as $F_{Y}(x, y)=\sum_{i} F^{i}(x, y) \frac{\partial}{\partial y^{i}}(y)$.

Note that Lemma 3.3 allows a one-to-one correspondence between $\frac{\partial}{\partial x^{i}}$ and $\frac{\partial}{\partial y^{i}}$ even across different charts, thus the transport operator is well defined. The following lemma (which is proved in local coordinates) is needed for the next theorem:

Lemma 3.14. Assume that $w^{\varepsilon}$ is a sequence in $W^{1,2}(M)$ which converges strongly to some $w \in W^{1,2}(M)$. Then

$$
\mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(w^{\varepsilon}\right) \rightarrow w \quad \text { strongly in } L^{2}(M \times Y), \quad \mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(\nabla_{M} w^{\varepsilon}\right) \rightarrow\left(\nabla_{M} w\right)_{Y} \quad \text { strongly in } L^{2}\left(M ; L^{2} T Y\right)
$$

We now present the main result:
Theorem 3.15. Let $\left\{w^{\varepsilon}\right\} \subset W^{1,2}(M)$ be a sequence such that $\left\|w^{\varepsilon}\right\|_{L^{2}(M)} \leqslant C$ and $\left\|\nabla_{M} w^{\varepsilon}\right\|_{L^{2} T M} \leqslant C$ with a constant $C$ independent of $\varepsilon$. Then there exist $a w \in W^{1,2}(M)$ and $a \hat{w} \in L^{2}\left(M ; W_{\#}^{1,2}(Y)\right)$ such that along a subsequence

$$
\mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(w^{\varepsilon}\right) \rightarrow w \quad \text { strongly in } L^{2}(M \times Y), \quad \mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(\nabla_{M} w^{\varepsilon}\right) \rightharpoonup\left(\nabla_{M} w\right)_{Y}+\nabla_{Y}^{(x)} \hat{w} \quad \text { weakly in } L^{2}\left(M ; L^{2} T Y\right)
$$

where by abuse of notation we use $\nabla_{Y}^{(x)} w$ to denote a function $(x, y) \mapsto \nabla_{Y}^{(x)} w(x, y)$.
Proof. By the usual compactness results, there exists a $w \in W^{1,2}(M)$ such that $w^{\varepsilon} \rightharpoonup w$ in $W^{1,2}(M)$. The compact embedding $W^{1,2}(M) \hookrightarrow L^{2}(M)$ together with Lemma 3.8 yields the first convergence statement. Since $\mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(\nabla_{M} w^{\varepsilon}\right)$ and $\left(\nabla_{M} w\right)_{Y}$ are bounded in $L^{2}\left(M ; L^{2} T Y\right)$, there exists a $\xi \in L^{2}\left(M ; L^{2} T Y\right)$ such that $\mathcal{T}_{\mathcal{A}}^{\varepsilon}\left(\nabla_{M} w^{\varepsilon}\right)-\left(\nabla_{M} w\right)_{Y} \rightharpoonup \xi$ in $L^{2}\left(M ; L^{2} T Y\right)$. By constructing a special test function, one can show that $\xi$ is orthogonal to functions $\psi \in L^{2}\left(M ; W_{\#}^{1,2} T Y\right)$ with $\operatorname{div}_{Y}^{(x)}(\psi)=0$. The result now follows as in the proof of Theorem 3.12.

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