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## Topology/Dynamical Systems

# Minimal sets of R-closed surface homeomorphisms

Ensembles minimaux des homéomorphismes R-fermés de surfaces

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#### ABSTRACT

Let *M* be an orientable connected closed surface and *f* be a homeomorphism on *M*. Suppose that the set  $\hat{E}_f := \{(x, y) \mid y \in \overline{O_f(x)}\}$  is closed. Then the suspension of *f* satisfies one of the following conditions: 1) the closure of each element of it is toral; 2) there is a minimal set which is not locally connected. Moreover, we show that if the set  $\hat{E}_f$  of a homeomorphism *f* on a compact metrizable space is closed, then so is  $\hat{E}_{f^k}$  for any natural number *k*.

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RÉSUMÉ

Soit *M* une surface fermée connexe orientable sans bord et *f* un homéomorphisme de *M*. Supposons que l'ensemble  $\hat{E}_f := \{f(x, y) \mid y \in \overline{O_f(x)}\}$  soit fermé. Alors la suspension vérifie l'une des conditions suivantes : 1) l'adhérence d'un élément est un tore ; 2) il existe un ensemble minimal qui n'est pas localement connexe. De plus, on démontre que si l'ensemble  $\hat{E}_f$  d'un homéomorphisme *f* d'un espace compact métrisable est fermé, alors les  $\hat{E}_{fk}$  sont fermés pour tout entier naturel *k*.

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### 1. Preliminaries

In [5] and [4], the authors have classified the minimal sets of torus homeomorphisms and their complements. On the other hand, for nontrivial (i.e. neither minimal nor identical) surface homeomorphisms, we show the existence of minimal sets which are not locally connected. Precisely, there is such a minimal set for a nontrivial aperiodic *R*-closed homeomorphism *f* on an orientable connected closed surface *M*, if *M* has genus  $\ge 2$  or if *f* is not "an irrational rotation". By closed surfaces, we mean compact 2-manifolds without boundaries. In addition, we show that any positive iteration of an *R*-closed homeomorphism on a compact metrizable space is *R*-closed.

For a subset *U* of a topological space, *U* is locally connected if every point of *U* admits a neighbourhood basis consisting of open connected subsets. For a (binary) relation *E* on a set *X* (i.e. a subset of  $X \times X$ ), let  $E(x) := \{y \in X \mid (x, y) \in E\}$  for an element *x* of *X*. For a subset *A* of *X*, we say that *A* is *E*-saturated if  $A = \bigcup_{x \in A} E(x)$ . Also *E* defines a relation  $\hat{E}$  on *X* with  $\hat{E}(x) = \overline{E(x)}$ . Recall that *E* is pointwise almost periodic if  $\hat{E}$  is an equivalence relation and *E* is *R*-closed if  $\hat{E}$  is closed. For an equivalence relation *E*, the collection of equivalence classes  $\{E(x) \mid x \in X\}$  is a decomposition of *X*, denoted by  $\mathcal{F}_E$ . By a decomposition, we mean a family  $\mathcal{F}$  of pairwise disjoint subsets of a set *X* such that  $X = \coprod \mathcal{F}$ . For a homeomorphism *f* 

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on *X*, let  $E_f$  be the equivalence relation defined by  $E_f(x) := \{f^k(x) | k \in \mathbb{Z}\}$ . The homeomorphism *f* is said to be *R*-closed if  $E_f$  is *R*-closed. Then *f* is *R*-closed if and only if  $\hat{E}_f = \{(x, y) | y \in \overline{O_f(x)}\}$  is closed. Note that *f* on a locally compact Hausdorff space is pointwise almost periodic if and only if  $\hat{E}_f$  is an equivalence relation (cf. Theorem 4.10 [3]). We say that an equivalence relation *E* is *L*-stable if for an element *x* of *X* and for any open neighbourhood *U* of  $\hat{E}(x)$ , there is an *E*-saturated open neighbourhood *V* of  $\hat{E}(x)$  contained in *U*. In [2], Erdös and Stone have shown the following: If a continuous mapping *f* of a topological space *X* in itself is either pointwise recurrent or pointwise almost periodic, then so is  $f^k$  for each positive integer *k*. Moreover Gottschalk and Hedlund have shown group action cases (see Theorems 2.24, 4.04, and 7.04 [3]). We show the following key lemma which is an *R*-closed version of this fact on a compact metrizable space.

#### **Lemma 1.1.** Let f be a homeomorphism on a compact metrizable space X. If f is R-closed, then so is $f^n$ for any $n \in \mathbb{Z}_{>0}$ .

**Proof.** Put  $E := E_f$  and  $E^n := E_{f^n}$ . By Corollary 1.3 [7], we have that  $\hat{E}$  is an equivalence relation and so f is pointwise almost periodic. Since f is pointwise almost periodic, by Theorem 1 [2], we have that  $f^n$  is also pointwise almost periodic. Then  $\hat{E}^n$  is an equivalence relation. By Corollary 3.6 [7], E is L-stable and it suffices to show that  $E^n$  is L-stable. Note that  $E^n(x) \subseteq E(x)$  and so  $\hat{E}^n(x) \subseteq \hat{E}(x)$ . For  $x \in X$  with  $\hat{E}^n(x) = \hat{E}(x)$  and for any open neighbourhood U of  $\hat{E}(x) = \hat{E}^n(x)$ , since E is L-stable, there is an E-saturated open neighbourhood V of  $\hat{E}(x)$  contained in U. Since  $E^n(x) \subseteq E(x)$ , we have that V is also an  $E^n$ -saturated open neighbourhood V of  $\hat{E}(x)$ . Fix any  $x \in X$  with  $\hat{E}(x) \neq \hat{E}^n(x)$ . Since minimal sets are distinct or coincident, put  $\{\hat{E}_1, \ldots, \hat{E}_k\} := \{\hat{E}(f^l(x)) \mid l = 0, 1, \ldots, n - 1\}$  such that  $\hat{E}_1 = \hat{E}^n(x)$  and  $\hat{E}_i \cap \hat{E}_j = \emptyset$  for any  $i \neq j \in \{1, \ldots, k\}$ . Let  $\hat{E}' = \hat{E}_2 \sqcup \cdots \sqcup \hat{E}_k$ . Then  $\hat{E}_1$  and  $\hat{E}'$  are closed and  $\hat{E} = \hat{E}_1 \sqcup \cdots \sqcup \hat{E}_k = \hat{E}_1 \sqcup \hat{E}'$ . For any sufficiently small  $\varepsilon > 0$ , let  $U_{1,\varepsilon} = B_{\varepsilon}(\hat{E}_1)$  (resp.  $U_{\varepsilon}' = B_{\varepsilon}(\hat{E}')$ ) be the open  $\varepsilon$ -neighbourhood of  $\hat{E}_1$  (resp.  $\hat{E}'$ ). Since  $\varepsilon$  is small and X is normal, we obtain  $U_{1,\varepsilon} \cap U_{\varepsilon}' = \emptyset$ ,  $\overline{U_{1,\varepsilon/2}} \subseteq U_{1,\varepsilon}$ , and  $\overline{U_{\varepsilon/2}} \subseteq U_{\varepsilon}'$ . Since E is L-stable, there are neighbourhoods  $V_{1,\varepsilon} \subseteq U_{1,\varepsilon/2}$  (resp.  $V_{\varepsilon}' \subseteq U_{\varepsilon/2}'$ ) of  $\hat{E}_1$  (resp.  $\hat{E}'$ ) such that  $V_{1,\varepsilon} \sqcup V_{\varepsilon}'$  is an E-saturated neighbourhood of  $\hat{E}(x)$ . Since  $\hat{V}_1, \hat{U} \lor_{\delta}'$  is  $f^n$ -invariant and  $U_{1,\varepsilon} \cap U_{\varepsilon}' = \emptyset$ , we obtain  $V_{1,\varepsilon} \sqcup V_{\delta}'$  is  $f^n(V_{1,\delta}) \sqsubseteq V_{1,\varepsilon} \sqcup V_{\delta}'$  is a  $f^n(V_{1,\delta}) \sqcup V_{\delta}' = f^n(V_{1,\delta})$  and  $f^n(V_{\delta}') \cap V_{1,\delta} = \emptyset$ . Then  $\hat{E}_1 = \hat{E}^n(x)$  with  $V_{1,\delta} \subseteq U_{1,\varepsilon} = B_{\varepsilon}(\hat{E}_1) = B_{\varepsilon}(\hat{E}_1) = B_{\varepsilon}(\hat{E}^n(x))$ . This implies that  $V_{1,\delta}$  is an  $E^n$ -saturated neighbourhood of  $\hat{E}_1 = \hat{E}^n(x)$  with  $V_{1,\delta} \subseteq U_{$ 

Note this lemma is not true for compact  $T_1$  spaces (e.g. a homeomorphism f on a non-Hausdorff 1-manifold  $X = \{0_-, 0_+\} \sqcup [0, 1]$  by  $f(0_{\pm}) = 0_{\pm}$  and  $f|_{[0, 1]} = id$ ).

## 2. Main results

From now on, let *M* be an orientable connected closed surface and *f* a nontrivial *R*-closed homeomorphism on *M* which is not periodic. We say that *f* on *S*<sup>2</sup> is a topological irrational rotation if there is an irrational number  $\theta_0 \in \mathbb{R} \setminus \mathbb{Q}$  such that *f* is topologically conjugate to a map on a unit sphere in  $\mathbb{R}^3$  with the cylindrical polar coordinates by  $(\rho, \theta, z) \rightarrow (\rho, \theta + \theta_0, z) \in \mathbb{R}_{\geq 0} \times S^1 \times \mathbb{R}$ . Also *f* on  $\mathbb{T}^2$  is a topological irrational rotation if there is an irrational number  $\theta_0 \in \mathbb{R} \setminus \mathbb{Q}$  such that some positive iteration of *f* is topologically conjugate to a map  $S^1 \times S^1 \rightarrow S^1 \times S^1$  by  $(\theta, \varphi) \rightarrow (\theta + \theta_0, \varphi)$ .

**Lemma 2.1.** If every minimal set is locally connected, then f is a topological irrational rotation on  $M = S^2$  or  $\mathbb{T}^2$ .

**Proof.** By Theorem 1 and Theorem 2 [1], since f is pointwise almost periodic, every orbit closure is a finite subset or a finite disjoint union of simple closed curves. We will show that there is a finite disjoint union of simple closed curves. Otherwise f is pointwise periodic. By [6], we get that f is periodic, which contradicts that f is not periodic. By Lemma 1.1, we have that  $f^n$  is also R-closed for any  $n \in \mathbb{Z}_{>0}$ . Hence there is a positive integer n such that  $f^n$  has a simple closed curve as a minimal set. Then Lemma 2.7 [8] implies that M is either  $\mathbb{T}^2$  or  $S^2$ . Moreover if  $M = S^2$  then n = 1. Suppose  $M = \mathbb{T}^2$  (resp.  $S^2$ ). Then the set  $\mathcal{F}_{\hat{E}_{f^n}}$  of orbit closures consists of essential circles (resp. two singular points and circles). For any invariant closed annulus A of  $f^n$  in M, let  $\gamma$  be a closed interval transverse to  $\mathcal{F}_{\hat{E}_{f^n}}|_A$  and connecting the boundaries of A. Then transverse intervals  $f^{nk}(\gamma) \subset A$  for any  $k \in \mathbb{Z}$  are pairwise disjoint and form a dense subset in A. Since each invariant circle in A has an irrational rotation number, by the definition of rotation number, the rotation number function on A is constant. This implies that the rotation number function of  $f^n$  on  $T^2$  (resp.  $S^2 - \text{Sing}(f)$ ) is constant and so that f is a topological irrational rotation.  $\Box$ 

This implies our main results:

**Theorem 2.2.** Let M be an orientable connected closed surface and f be a nontrivial R-closed homeomorphism on M which is not periodic. Then one of the following holds:

- 1) *f* is a topological irrational rotation.
- 2) There is a minimal set which is not locally connected.

*Moreover* 2) *holds when* M *has genus*  $\geq$  2.

Taking a suspension, we have the following corollary:

**Corollary 2.3.** Let M be an orientable connected closed surface and f be an R-closed homeomorphism on M. Then the suspension of f satisfies one of the following conditions:

- 1) The closure of each element of it is toral.
- 2) There is a minimal set which is not locally connected.

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