



Mathematical Analysis/Calculus of Variations

## 3D–2D dimensional reduction for a nonlinear optimal design problem with perimeter penalization

*Réduction dimensionnelle 3D–2D d'un problème non linéaire d'optimisation de forme avec pénalisation sur le périmètre*

Graça Carita<sup>a</sup>, Elvira Zappale<sup>b</sup><sup>a</sup> CIMA-UE, Departamento de Matemática, Universidade de Évora, Rua Romão Ramalho, 59 7000-671 Évora, Portugal<sup>b</sup> D.I.I.N., Università degli Studi di Salerno, Via Ponte Don Melillo, 84084 Fisciano (SA), Italy

## ARTICLE INFO

## Article history:

Received 24 January 2012

Accepted after revision 9 November 2012

Available online 23 November 2012

Presented by Philippe G. Ciarlet

## ABSTRACT

A 3D–2D dimension reduction for a nonlinear optimal design problem with a perimeter penalization is performed in the realm of  $\Gamma$ -convergence, providing an integral representation for the limit functional.

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## R É S U M É

On effectue dans ce travail une réduction dimensionnelle 3D–2D d'un problème non linéaire d'optimisation de forme avec une pénalisation du périmètre. Une représentation intégrale de la fonctionnelle limite est obtenue.

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## Version française abrégée

On s'intéresse dans ce travail au comportement asymptotique d'une suite de problèmes non linéaires d'optimisation de forme avec pénalisation du périmètre sur le domaine cylindrique  $\Omega(\varepsilon) := \omega \times (-\varepsilon, \varepsilon)$ , où  $\varepsilon > 0$  et  $\omega$  est un ouvert borné de  $\mathbb{R}^2$ . On suppose que le domaine  $\Omega(\varepsilon)$ , occupé par le solide, est constitué par deux matériaux hyperélastiques dont les densités d'énergie,  $W_i : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ ,  $i = 1, 2$ , sont continues, vérifiant la condition de croissance

$$\beta'(|F|^p - 1) \leq W_i(F) \leq \beta(1 + |F|^p) \quad \text{pour tout } F \in \mathbb{R}^{3 \times 3}, p > 1, i = 1, 2, \text{ avec } \beta \geq \beta' > 0. \quad (1)$$

Plus précisément, on considère le problème de minimisation suivant :

$$\inf_{\substack{v \in W^{1,p}(\Omega(\varepsilon); \mathbb{R}^3) \\ \chi_{E(\varepsilon)} \in BV(\Omega(\varepsilon); \{0,1\})}} \left\{ \frac{1}{\varepsilon} \left( \int_{\Omega(\varepsilon)} (\chi_{E(\varepsilon)} W_1 + (1 - \chi_{E(\varepsilon)}) W_2) (\nabla v) \, dx - \int_{\Omega(\varepsilon)} f_\varepsilon \cdot v \, dx + P(E(\varepsilon); \Omega(\varepsilon)) \right) \right. \\ \left. v = 0 \text{ sur } \partial\omega \times (-\varepsilon, \varepsilon), \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)} \, dx = \lambda \right\}, \quad (2)$$

E-mail addresses: gcarita@uevora.pt (G. Carita), ezappale@unisa.it (E. Zappale).

où  $E(\varepsilon) \subset \Omega(\varepsilon)$  est un ensemble mesurable de périmètre fini et  $f_\varepsilon \in L^{p'}(\Omega(\varepsilon); \mathbb{R}^3)$  où  $\frac{1}{p} + \frac{1}{p'} = 1$  et  $\lambda \in [0, 1]$  est la fraction du volume rempli par le premier matériau.

Nous commençons par effectuer un changement de variables afin de rendre le domaine indépendant de  $\varepsilon$  (cf. (7)). On obtient ainsi le problème suivant

$$\inf_{\substack{u \in W^{1,p}(\Omega; \mathbb{R}^3) \\ \chi \in BV(\Omega; \{0,1\})}} \left\{ \int_{\Omega} (\chi W_1 + (1 - \chi)W_2) \left( \nabla_\alpha u \Big|_{\frac{1}{\varepsilon}} \nabla_3 u \right) dx - \int_{\Omega} f \cdot u dx + \left| \left( D_\alpha \chi \Big|_{\frac{1}{\varepsilon}} D_3 \chi \right) \right|(\Omega) : \right. \\ \left. u = 0 \text{ sur } \partial\omega \times (-1, 1), \frac{1}{\mathcal{L}^3(\Omega)} \int_{\Omega} \chi dx = \lambda \right\}. \tag{3}$$

En utilisant les techniques de  $\Gamma$ -convergence, on démontre que, lorsque  $\varepsilon$  tend vers zéro, le problème (3) converge vers un problème non linéaire bidimensionnel.

Il suit un résultat général où la pénalisation du périmètre dans le problème initial est remplacée par une intégrale elliptique. Ainsi, on considère  $\Psi : \mathbb{R}^3 \rightarrow [0, +\infty[$  une fonction paire, continue, positivement homogène de degré 1 et telle que

$$\exists C \in ]0, +\infty[ : \forall v \in \mathbb{R}^3 \quad \frac{1}{C} |v| \leq \Psi(v) \leq C|v|. \tag{4}$$

On étudie donc, le comportement asymptotique, lorsque  $\varepsilon$  tend vers zéro, du problème suivant

$$\inf_{\substack{v \in W^{1,p}(\Omega(\varepsilon); \mathbb{R}^3) \\ \chi_{E(\varepsilon)} \in BV(\Omega(\varepsilon); \{0,1\})}} \left\{ \frac{1}{\varepsilon} \left( \int_{\Omega(\varepsilon)} (\chi_{E(\varepsilon)} W_1 + (1 - \chi_{E(\varepsilon)})W_2) (\nabla v) dx - \int_{\Omega(\varepsilon)} f_\varepsilon \cdot v dx + \int_{\partial E(\varepsilon)} \Psi(v_{E(\varepsilon)}) d\mathcal{H}^2 \right) : \right. \\ \left. v = 0 \text{ sur } \partial\omega \times (-\varepsilon, \varepsilon), \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)} dx = \lambda \right\}, \tag{5}$$

où  $\mathcal{H}^2$  désigne la mesure de Hausdorff restreinte à  $\partial E(\varepsilon)$  et  $v_{E(\varepsilon)}$  est la normale extérieure à  $E(\varepsilon)$ . On obtient finalement, le problème limite (19).

### 1. Introduction and setting of the problem

The study of thin structures has been the object of many investigations. In particular, in mechanical engineering it is important for applications to minimize, under a given system of loads, the compliance (namely, the opposite of the total energy at equilibrium) of a given structure, satisfying a constraint on the volume. In order to design thin structures with the best possible resistance-weight ratio, the asymptotic behaviour of the compliance as the thickness of the sample tends to zero is studied. For a background on the modelling of thin plates we refer to the monographs of [1] and [11].

Let  $\Omega(\varepsilon) := \omega \times (-\varepsilon, \varepsilon)$ , where  $\omega$  is a bounded open domain of  $\mathbb{R}^2$  and  $\varepsilon > 0$ , and for the sake of illustration let us assume that  $\Omega(\varepsilon)$  is clamped on its lateral boundary. We suppose also that  $\Omega(\varepsilon)$  is filled with two materials with respective energy densities  $W_1$  and  $W_2$ , where  $W_i : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ ,  $i = 1, 2$  are continuous functions satisfying (1) (see Remark 2 below, where assumption (1) is discussed).

Let  $E(\varepsilon)$  be the first phase,  $f_\varepsilon$  be the given load on  $\Omega(\varepsilon)$  and assume that the volume fraction of each phase is given by  $\lambda := \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)}(x) dx \in [0, 1]$ , where  $\chi_{E(\varepsilon)}$  denotes the characteristic function of the phase  $E(\varepsilon)$ . The compliance  $C^\varepsilon(\chi_{E(\varepsilon)})$  is defined as

$$C^\varepsilon(\chi_{E(\varepsilon)}) := - \inf_{v \in W^{1,p}(\Omega(\varepsilon); \mathbb{R}^3)} \left\{ \frac{1}{\varepsilon} \int_{\Omega(\varepsilon)} ((\chi_{E(\varepsilon)} W_1 + (1 - \chi_{E(\varepsilon)})W_2) (\nabla v) - f_\varepsilon \cdot v) dx : v = 0 \text{ on } \partial\omega \times (-\varepsilon, \varepsilon) \right\}.$$

In [6–8] the asymptotic behaviour of a 3D optimal elastic compliance problem is studied, as the thickness (or the cross section in the case of beams) tends to zero and the volume fraction in the design region remains unchanged. It is assumed that the material has a convex and 2-homogeneous potential and the analysis is performed in the small-displacement setting. The prescription of the volume in the minimum problem can be dropped by adding a Lagrange multiplier to penalize the volume in the cost functional. The asymptotic analysis performed in these papers leads to a fictitious material with local density, taking all the values in  $[0, 1]$  and not to a precise limit set, due to the loss of compactness in the characteristic functions.

In this work we focus our attention on studying the worst possible design of a two-phase mixture of elastic materials in a thin film in the same spirit of [13] and [9], where the asymptotic analysis of a two-field minimization problem has been studied (i.e.  $(\chi, u)$  (design region, deformation)) as the thickness of the sample tends to zero. Having in mind the results contained in [2] and [17], we introduce a perimeter penalization in our functional in order to derive from the 3D energy a limiting 2D model where the design region is explicitly determined, and we refer to [10] for a detailed study about regularity of the limits (set, deformation).

Let us consider the optimal design problem in (2) where  $E(\varepsilon) \subset \Omega(\varepsilon)$  is a measurable subset of  $\Omega(\varepsilon)$  with finite perimeter, i.e.,

$$P(E(\varepsilon); \Omega(\varepsilon)) := \sup \left\{ \int_{E(\varepsilon)} \operatorname{div} \varphi \, dx : \varphi \in C_c^1(\Omega(\varepsilon); \mathbb{R}^3), \|\varphi\|_{L^\infty} \leq 1 \right\} < +\infty, \tag{6}$$

and we assume that the load  $f_\varepsilon \in L^{p'}(\Omega(\varepsilon); \mathbb{R}^3)$ , with  $\frac{1}{p} + \frac{1}{p'} = 1$ .

In order to study the asymptotic behaviour of (2) we reformulate our problem in a fixed 3D domain through a  $\frac{1}{\varepsilon}$ -dilation in the transverse direction  $x_3$  and then we perform  $\Gamma$ -convergence with respect to the pair (design region, deformation). Set

$$\begin{aligned} \Omega &:= \omega \times (-1, 1), & E_\varepsilon &:= \{(x_1, x_2, x_3) \in \Omega : (x_1, x_2, \varepsilon x_3) \in E(\varepsilon)\}, \\ u(x_1, x_2, x_3) &:= v(x_1, x_2, \varepsilon x_3), & f(x_1, x_2, x_3) &:= f_\varepsilon(x_1, x_2, \varepsilon x_3), & \chi_{E_\varepsilon}(x_1, x_2, x_3) &:= \chi_{E(\varepsilon)}(x_1, x_2, \varepsilon x_3), \end{aligned} \tag{7}$$

where  $v$  is any admissible field for (2).

In the sequel we will denote  $x_\alpha := (x_1, x_2)$ ,  $dx_\alpha := dx_1 dx_2$  and  $\nabla_\alpha$  and  $D_\alpha$  will be identified with the pair  $(\nabla_1, \nabla_2)$ ,  $(D_1, D_2)$ , respectively. For every matrix  $\bar{F} \in \mathbb{R}^{3 \times 2}$  and any  $z \in \mathbb{R}^3$ ,  $F := (\bar{F}|z)$  represents the matrix in  $\mathbb{R}^{3 \times 3}$  whose first two columns are those of  $\bar{F}$  and the last column is given by the vector  $z$ .

Observe that by (6) and using the definition of total variation,  $P(E_\varepsilon; \Omega(\varepsilon)) = |D\chi_{E_\varepsilon}|(\Omega(\varepsilon))$ . Making the change of variables  $y_3 := \varepsilon x_3$  and  $y_\alpha := x_\alpha$  we have  $\frac{1}{\varepsilon}|D\chi_{E_\varepsilon}|(\Omega(\varepsilon)) = |(D_\alpha \chi_\varepsilon | \frac{1}{\varepsilon} D_3 \chi_\varepsilon)|(\Omega)$ , where  $\chi_\varepsilon := \chi_{E_\varepsilon}$  stands for the characteristic function of  $E_\varepsilon$ . Hence we are led to the rescaled minimum problem (3).

For every  $\varepsilon > 0$ , let  $J_\varepsilon : L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$  be the functional defined as follows

$$J_\varepsilon(\chi, u) := \begin{cases} \int_\Omega (\chi W_1(\nabla_\alpha u | \frac{1}{\varepsilon} \nabla_3 u) + (1 - \chi) W_2(\nabla_\alpha u | \frac{1}{\varepsilon} \nabla_3 u)) \, dx - \int_\Omega f \cdot u \, dx + |(D_\alpha \chi | \frac{1}{\varepsilon} D_3 \chi)|(\Omega) \\ \text{in } BV(\Omega; \{0, 1\}) \times W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty \text{ otherwise.} \end{cases} \tag{8}$$

Let  $V : \{0, 1\} \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty]$  be given by

$$V(\chi, F) := \chi W_1(F) + (1 - \chi) W_2(F), \tag{9}$$

with  $W_1$  and  $W_2$  satisfying (1). Analogously, let  $\bar{V} : \{0, 1\} \times \mathbb{R}^{3 \times 2} \rightarrow [0, +\infty]$  be defined as

$$\bar{V}(\chi, \bar{F}) := \chi \bar{W}_1(\bar{F}) + (1 - \chi) \bar{W}_2(\bar{F}), \quad \text{with } \bar{W}_i(\bar{F}) := \inf_{c \in \mathbb{R}^3} W_i(\bar{F}|c), \quad \bar{F} \in \mathbb{R}^{3 \times 2}, \quad i = 1, 2. \tag{10}$$

Consider the functional  $J_0 : L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$  as

$$J_0(\chi, u) := \begin{cases} 2 \int_\omega Q \bar{V}(\chi, \nabla_\alpha u) \, dx_\alpha - \int_{-1}^1 \int_\omega f \cdot u \, dx_\alpha \, dx_3 + 2|D_\alpha \chi|(\omega), \\ \text{if } (\chi, u) \in BV(\omega; \{0, 1\}) \times W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty \text{ otherwise,} \end{cases} \tag{11}$$

where  $Q \bar{V}$  stands for the quasiconvexification of  $\bar{V}$  in the second variable. Namely, for every  $(\chi, \bar{F}) \in \{0, 1\} \times \mathbb{R}^{3 \times 2}$

$$Q \bar{V}(\chi, \bar{F}) := \inf_{Q'} \left\{ \int_{Q'} \bar{V}(\chi, \bar{F} + \nabla_\alpha \varphi) \, dx_\alpha : \varphi \in W_0^{1,p}(Q'; \mathbb{R}^3) \right\}, \tag{12}$$

where  $Q' \subset \mathbb{R}^2$  denotes the unit cube.

Our main result is the following:

**Theorem 1.1.** *The family of functionals  $\{J_\varepsilon\}$   $\Gamma$ -converges, with respect to the strong topology of  $L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)$  to  $J_0$ , as  $\varepsilon \rightarrow 0^+$ .*

**Remark 1.** We observe that Theorem 1.1 entails the convergence, as  $\varepsilon \rightarrow 0^+$ , of problems (2) in their rescaled version (3) to the problem

$$\inf_{\substack{u \in W_0^{1,p}(\omega; \mathbb{R}^3) \\ \chi \in BV(\omega; \{0, 1\})}} \left\{ 2 \int_\omega Q \bar{V}(\chi, \nabla_\alpha u) \, dx_\alpha - \int_{-1}^1 \int_\omega f \cdot u \, dx_\alpha \, dx_3 + 2|D_\alpha \chi|(\omega) : \frac{1}{\mathcal{L}^2(\omega)} \int_\omega \chi \, dx_\alpha = \frac{1}{2} \lambda \right\}. \tag{13}$$

In fact, this is due to the strong convergence in  $L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)$ , of the sequence of almost minimizers  $\{(\chi_\varepsilon, u_\varepsilon)\}$  of (3) to  $(\chi, u) \in BV(\omega; \{0, 1\}) \times W_0^{1,p}(\omega; \mathbb{R}^3)$ . And so, the constraint in the volume fraction  $\frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)} \, dx = \frac{1}{\mathcal{L}^3(\Omega)} \int_\Omega \chi_\varepsilon \, dx = \lambda$  is kept in the limit, as well as the boundary conditions (cf. Remark 2).

It is worthwhile to compare Theorem 1.1 with similar results in [13] and [9]. To this aim we observe that, in spite of what is proven therein, namely  $\Gamma$ -convergence results with respect to the convergence  $L^\infty_{\text{weak}^*} \times L^p$  for  $(\chi, u)$ , the presence of the perimeter in our energy (8), allows us to have a stronger convergence on the characteristic functions and thus to determine the worst possible design set. On the other hand, the fact that the perimeter is inserted in our model leads naturally to compare our results with those contained in [2]. Indeed, if  $W_1$  and  $W_2$  are of type  $W_1(\cdot) := \alpha'|\cdot|^2$  and  $W_2(\cdot) := \alpha|\cdot|^2$ , with  $0 < \alpha' < \alpha$  suitable constants, clearly  $Q\bar{V}(\chi, \bar{F})$  coincides with  $\alpha'\chi|\bar{F}|^2 + \alpha(1 - \chi)|\bar{F}|^2$ . Hence, by [2, Theorem 2.2] the solution of the minimum problem (13) is locally Hölder continuous and the optimal design set is equivalent to an open set  $A \times (-1, 1)$ ,  $A \subset \omega$ . More refined results, about regularity in 2D, in the convex setting, may be found in [14,15] and in the references quoted therein.

In our model, the lack of convexity in  $W_1$  and  $W_2$  entails, as underlined by Proposition 2.2, that we obtain a limit energy which depends continuously on the characteristic function of the design set and requires a quasiconvexification procedure in the deformation variable. We refer to [10] for regularity results related to our setting.

Details about the results are contained in the next section, while for the properties related to  $\Gamma$ -convergence, sets of finite perimeter and BV functions we refer to [12] and [3], respectively.

**2. The limit problem**

We start by stating the properties of the energy densities in (9) and (10) that we will exploit in the sequel.

**Proposition 2.1.** *Let  $\bar{V}$  be as in (10). Then  $\bar{V}$  is continuous and satisfies*

$$\beta'(|\bar{F}|^p - 1) \leq \bar{V}(\chi, \bar{F}) \leq \beta(1 + |\bar{F}|^p), \tag{14}$$

where  $\beta'$  and  $\beta$  are the constants in (1). Moreover,  $|\bar{V}(\chi, \bar{F}) - \bar{V}(\chi', \bar{F})| \leq 2\beta|\chi - \chi'|(1 + |\bar{F}|^p)$ .

**Proposition 2.2.** *The function  $Q\bar{V}$  in (12) is continuous and satisfies (14), and*

$$|Q\bar{V}(\chi, \bar{F}) - Q\bar{V}(\chi', \bar{F})| \leq C|\chi' - \chi|(1 + |\bar{F}|^p). \tag{15}$$

**Remark 2.** We claim that energy bounded sequences  $\{(\chi_\varepsilon, u_\varepsilon)\}$  for problem (3), with  $u_\varepsilon$  clamped on  $\partial\omega \times (-1, 1)$ , are compact in  $L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)$  and with limit in  $L^1(\omega; \{0, 1\}) \times L^p(\omega; \mathbb{R}^3)$ .

If  $\{(\chi_\varepsilon, u_\varepsilon)\}$  is a sequence such that  $J_\varepsilon(\chi_\varepsilon, u_\varepsilon) \leq C$ , then there exists  $C' \in \mathbb{R}^+$  such that the following bounds hold

$$\|u_\varepsilon\|_{W^{1,p}} \leq C', \quad \left\| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon \right\|_{L^p} \leq C', \quad \left| \left( D_\alpha \chi_\varepsilon \Big| \frac{1}{\varepsilon} D_3 \chi_\varepsilon \right) (\Omega) \right| \leq C'.$$

An argument entirely similar to that exploited in [16, Lemma 3], entails that there exists  $u \in W^{1,p}(\Omega; \mathbb{R}^3)$  such that  $\nabla_3 u \equiv 0$ , and so  $u$  can be identified with a function (still denoted in the same way)  $u \in W^{1,p}(\omega; \mathbb{R}^3)$ . Thus we may find a subsequence, not relabelled,  $\{u_\varepsilon\}$  such that  $u_\varepsilon \rightharpoonup u$  in  $W^{1,p}(\Omega; \mathbb{R}^3)$ , and a measurable set  $E \subset \Omega$  such that  $\chi_\varepsilon \rightharpoonup^* \chi_E$  and  $D_3 \chi_E \equiv 0$ . Hence, there exists  $E' \subset \omega$ , with  $|D\chi_E|(\Omega) = 2|D\chi_{E'}|(\omega)$ , where  $E = E' \times (-1, 1)$ . In the following we will identify the set  $E$  with the set  $E'$  and denote  $\chi_{E'}$  by  $\chi$ .

**Proof of Theorem 1.1.** For every  $\varepsilon > 0$ , let  $J_\varepsilon$  be the functional in (8). The  $\Gamma$ -convergence with respect to the separable metric space  $L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)$  ensures that for each sequence  $\{\varepsilon\}$  there exists a subsequence, still denoted by  $\{\varepsilon\}$ , such that  $\Gamma - \lim_{\varepsilon \rightarrow 0^+} (L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)) J_\varepsilon$  exists.

For every  $(\chi, u) \in L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)$ , let  $J(\chi, u)$  be its  $\Gamma$ -limit. By virtue of Urysohn property, it suffices to prove that any sequence  $\{J_\varepsilon\}$  admits a further subsequence whose  $\Gamma$ -limit,  $J(\chi, u)$ , coincides with  $J_0(\chi, u)$  in (11).

We observe that if  $(\chi, u) \in (L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)) \setminus (BV(\omega; \{0, 1\}) \times W^{1,p}(\omega; \mathbb{R}^3))$ , then  $J(\chi, u) = +\infty$ . Indeed, if this is not the case, from  $J(\chi, u) < +\infty$  we would get the existence of a sequence  $\{(\chi_\varepsilon, u_\varepsilon)\}$  converging to  $(\chi, u)$  such that  $J_\varepsilon(\chi_\varepsilon, u_\varepsilon) < +\infty$  and by Remark 2 this would imply  $(\chi, u) \in BV(\omega; \{0, 1\}) \times W^{1,p}(\omega; \mathbb{R}^3)$ .

The remaining proof is divided into two steps. First we show the lower bound, then we prove the upper bound.

*Step one:* We claim that for every  $(\chi, u) \in BV(\omega; \{0, 1\}) \times W^{1,p}(\omega; \mathbb{R}^3)$

$$J(\chi, u) \geq 2 \int_{\omega} Q\bar{V}(\chi(x_\alpha), \nabla_\alpha u(x_\alpha)) dx_\alpha - \int_{-1}^1 \int_{\omega} f(x_\alpha, x_3) u(x_\alpha) dx_\alpha dx_3 + 2|D_\alpha \chi|(\omega).$$

To prove the claim, let  $\{(\chi_\varepsilon, u_\varepsilon)\} \subset L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)$  be a sequence converging to  $(\chi, u) \in BV(\omega; \{0, 1\}) \times W^{1,p}(\omega; \mathbb{R}^3)$ . For the forces and for the perimeter the lower bound follows by  $L^p$  strong convergence of  $\{u_\varepsilon\}$  and lower semicontinuity of the perimeter, respectively.

For what concerns the bulk energy, by virtue of the Decomposition Lemma for scaled gradients (cf. [5, Theorem 1.1]) there exist a subsequence of  $\{u_\varepsilon\}$ , not relabelled, and a sequence  $\{w_\varepsilon\}$  converging to  $u \in W^{1,p}(\omega; \mathbb{R}^3)$ , such that the scaled

gradients  $\{(\nabla_\alpha w_\varepsilon, \frac{1}{\varepsilon} \nabla_3 w_\varepsilon)\}$  are  $p$ -equiintegrable, and  $\mathcal{L}^3(\Omega \setminus A_\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0^+$ , where  $A_\varepsilon := \{x \in \Omega : u_\varepsilon \equiv w_\varepsilon\}$ . Denoting the bulk energy density of  $J_\varepsilon$  by  $V$  as in (9), one obtains

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0^+} \int_{\Omega} V\left(\chi_\varepsilon, \left(\nabla_\alpha u_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon\right)\right) dx \\ & \geq \liminf_{\varepsilon \rightarrow 0^+} \int_{\Omega} V\left(\chi_\varepsilon, \left(\nabla_\alpha w_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 w_\varepsilon\right)\right) dx - \beta \limsup_{\varepsilon \rightarrow 0^+} \int_{\Omega \setminus A_\varepsilon} \left(1 + \left|\left(\nabla_\alpha w_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 w_\varepsilon\right)\right|^p\right) dx \\ & \geq \liminf_{\varepsilon \rightarrow 0^+} \int_{\Omega} V\left(\chi_\varepsilon, \left(\nabla_\alpha w_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 w_\varepsilon\right)\right) dx \geq \liminf_{\varepsilon \rightarrow 0^+} \int_{\Omega} \bar{V}(\chi_\varepsilon, \nabla_\alpha w_\varepsilon) dx \geq \liminf_{\varepsilon \rightarrow 0^+} \int_{\Omega} Q \bar{V}(\chi_\varepsilon, \nabla_\alpha w_\varepsilon) dx. \end{aligned} \tag{16}$$

Observe that, by (15)

$$\int_{\Omega} |Q \bar{V}(\chi_\varepsilon, \nabla_\alpha w_\varepsilon) - Q \bar{V}(\chi, \nabla_\alpha w_\varepsilon)| dx \leq C \int_{\Omega} |\chi_\varepsilon - \chi| (1 + |\nabla_\alpha w_\varepsilon|^p) dx. \tag{17}$$

Thus, the  $p$ -equiintegrability of  $\{(\nabla_\alpha w_\varepsilon | \frac{1}{\varepsilon} \nabla_3 w_\varepsilon)\}$  and (17) ensure that as  $\varepsilon \rightarrow 0^+$ ,  $\chi_\varepsilon$  can be replaced by  $\chi$  in the right-hand side of (16).

The density  $Q \bar{V}(\chi(x_\alpha), \cdot)$  is quasiconvex in  $M^{3 \times 2}$  for a.e.  $x \in \Omega$ . Using an argument similar to that exploited in [16, Proposition 6] one concludes that  $Q \bar{V}(\chi(x_\alpha), \cdot)$  is quasiconvex also in  $M^{3 \times 3}$ . Thus, by the growth condition of  $Q \bar{V}$ , as stated in Proposition 2.2, the functional  $v \in W^{1,p}(\Omega; \mathbb{R}^3) \mapsto \int_{\Omega} Q \bar{V}(\chi(x_\alpha), \nabla_\alpha v(x)) dx$  is sequentially weakly lower semi-continuous with respect to  $W^{1,p}$ -weak topology. Hence,

$$\liminf_{\varepsilon \rightarrow 0^+} \int_{\Omega} Q \bar{V}(\chi, \nabla_\alpha w_\varepsilon) dx \geq 2 \int_{\omega} Q \bar{V}(\chi, \nabla_\alpha u) dx_\alpha.$$

By the superadditivity of the liminf we achieve the claim.

*Step two:* To prove the reverse inequality we start by observing that, fixing  $\chi \in BV(\omega; \{0, 1\})$ ,  $J(\chi, u) \leq \liminf_{\varepsilon \rightarrow 0^+} J_\varepsilon(\chi, u_\varepsilon)$  for every  $\{u_\varepsilon\} \subseteq L^p(\Omega; \mathbb{R}^3)$  and  $u \in W^{1,p}(\omega; \mathbb{R}^3)$  such that  $u_\varepsilon \rightarrow u$  in  $L^p(\Omega; \mathbb{R}^3)$ .

Thus it suffices to study the asymptotic behaviour with respect to the  $W^{1,p}$ -weak convergence of

$$\int_{\Omega} \left(\chi W_1\left(\nabla_\alpha u_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon\right) + (1 - \chi) W_2\left(\nabla_\alpha u_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon\right)\right) dx - \int_{\Omega} f \cdot u_\varepsilon dx.$$

Since  $\chi$  is fixed we can rewrite  $\chi W_1(\cdot) + (1 - \chi) W_2(\cdot)$  as a new function with explicit dependence on  $x_\alpha$ .

Namely, let  $W : \omega \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$  be given by

$$W(x_\alpha, F) := V(\chi(x_\alpha), F) = \chi(x_\alpha) W_1(F) + (1 - \chi(x_\alpha)) W_2(F),$$

for every  $(x_\alpha, F) \in \omega \times \mathbb{R}^{3 \times 3}$ .

Clearly,  $W$  is a Carathéodory function satisfying the growth condition  $\frac{1}{C} |F|^p - C \leq W(x_\alpha, F) \leq C(1 + |F|^p)$  for a.e.  $x_\alpha \in \omega$  and for all  $F \in \mathbb{R}^{3 \times 3}$ . Applying [4, Theorem 2.3] to the sequence of functionals  $\{G_\varepsilon\}$ , where  $G_\varepsilon : L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty)$  is given by

$$G_\varepsilon(u) := \begin{cases} \int_{\Omega} W(x_\alpha, (\nabla_\alpha u, | \frac{1}{\varepsilon} \nabla_3 u)) dx - \int_{\Omega} f \cdot u dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

and arguing as in [9, Remark 3.3] we get

$$\lim_{\varepsilon \rightarrow 0^+} \left( \int_{\Omega} W\left(x_\alpha, \left(\nabla_\alpha u_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon\right)\right) dx - \int_{\Omega} f \cdot u_\varepsilon dx \right) \leq 2 \int_{\omega} Q \bar{W}(x_\alpha, \nabla_\alpha u) dx_\alpha - \int_{-1}^1 \int_{\omega} f \cdot u dx_\alpha dx_3,$$

where  $\bar{W} : \omega \times \mathbb{R}^{3 \times 2}$  is defined by  $\bar{W}(x_\alpha, \bar{F}) := \inf_{c \in \mathbb{R}^3} W(x_\alpha, (\bar{F} | c))$ , and  $Q \bar{W}$  stands for the quasiconvexification of  $\bar{W}$  in the second variable.

Observing that by (10)

$$\bar{W}(x_\alpha, \bar{F}) = \chi(x_\alpha) \bar{W}_1(\bar{F}) + (1 - \chi(x_\alpha)) \bar{W}_2(\bar{F}) = \bar{V}(\chi(x_\alpha), \bar{F}), \quad \text{and} \quad Q \bar{W}(x_\alpha, \bar{F}) = Q \bar{V}(\chi(x_\alpha), \bar{F})$$

for every  $(x_\alpha, \bar{F}) \in \omega \times \mathbb{R}^{3 \times 2}$ , the proof is concluded.  $\square$

In the following we apply the previous analysis, with small changes, to the case where the perimeter penalization in (2) is replaced by a more general elliptic integral, such as in [17]. Namely, we consider  $\Psi : \mathbb{R}^3 \rightarrow [0, +\infty)$  even, continuous, positively 1-homogeneous, and satisfying (4).

Recall problem (5) and observe that by  $\int_{\partial E(\varepsilon)} \Psi(v_{E(\varepsilon)}) d\mathcal{H}^2$  we mean the integral with respect to the Hausdorff measure concentrated on  $\partial E(\varepsilon)$ , identified with  $S_{\chi E(\varepsilon)}$ , with exterior normal  $v_{E(\varepsilon)}$ .

By performing the same rescaling as in (7) we obtain the following formulation of (5) in the fixed domain  $\Omega$ ,

$$\inf_{\substack{u \in W^{1,p}(\Omega; \mathbb{R}^3) \\ \chi \in BV(\Omega; \{0,1\})}} \left\{ \int_{\Omega} (\chi W_1 + (1 - \chi) W_2) \left( \nabla_{\alpha} u \Big|_{\frac{1}{\varepsilon}} \nabla_3 u \right) dx - \int_{\Omega} f \cdot u dx + \int_{S_{\chi}} \Psi \left( v_{\alpha} \Big|_{\frac{1}{\varepsilon}} v_3 \right) d\mathcal{H}^2 : \right. \\ \left. u = 0 \text{ on } \partial\omega \times (-1, 1), \frac{1}{\mathcal{L}^3(\Omega)} \int_{\Omega} \chi dx = \lambda \right\}, \quad (18)$$

where  $\chi$  denotes the characteristic function of  $E_{\varepsilon}$ , and  $v$  the normal to its jump set,  $S_{\chi}$ .

Next we will identify functions defined in  $\mathbb{R}^3$  (or  $\mathbb{R}^2$ ) and their restrictions to  $S^2$  (or  $S^1$ ), and so the same notations will be adopted.

Let  $\bar{\Psi} : \mathbb{R}^2 \rightarrow [0, +\infty)$  be the function given by  $\bar{\Psi}(\eta) := \inf\{\Psi(\eta, \xi) : \xi \in \mathbb{R}\}$ , with  $\Psi : \mathbb{R}^3 \rightarrow [0, +\infty)$  as in (4).

Consider the following minimum problem

$$\inf_{\substack{u \in W_0^{1,p}(\omega; \mathbb{R}^3) \\ \chi \in BV(\omega; \{0,1\})}} \left\{ 2 \int_{\omega} Q \bar{V}(\chi, \nabla_{\alpha} u) dx_{\alpha} - \int_{-1}^1 \int_{\omega} f \cdot u dx_{\alpha} dx_3 + 2 \int_{S_{\chi}} \bar{\Psi}^{**}(v_{\alpha}) d\mathcal{H}^1 : \frac{1}{\mathcal{L}^2(\omega)} \int_{\omega} \chi dx_{\alpha} = \frac{1}{2} \lambda \right\}, \quad (19)$$

where  $\bar{\Psi}^{**}$  denotes the convex envelope of  $\bar{\Psi}$ . Namely,  $\bar{\Psi}^{**}(v) := \sup\{g : \mathbb{R}^2 \rightarrow \mathbb{R} : g \text{ is convex } g(v) \leq \bar{\Psi}(v) \forall v \in \mathbb{R}^2\}$ .

**Theorem 2.1.** Consider the problems (18), and their minimizers. Then the latter converge, with respect to the strong topology of  $L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)$ , to the minimum of problem (19).

We conclude by observing that the density  $\bar{\Psi}^{**}$  satisfies all the well established properties for the lower semicontinuity of surface integrals, such as *BV*-ellipticity (cf. [3, Definition 5.13 and Theorem 5.14]), since any continuous even function  $\phi : S^{N-1} \rightarrow [0, +\infty)$  is *BV*-elliptic if and only if its positive 1-homogeneous extension is convex.

## Acknowledgements

The authors thank Irene Fonseca for having suggested this problem and they acknowledge the support and the kind hospitality of Departamento de Matemática, Universidade de Évora and DIEII at Università di Salerno. The work of both authors was partially supported by CIMA-UE, financed by FCT (Fundação para a Ciência e Tecnologia) and GNAMPA through Project “Problemi variazionali e misure di Young nella meccanica dei Materiali Complessi”.

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