



Partial Differential Equations/Mathematical Problems in Mechanics

Hölder stability estimates for some inverse pointwise source problems

Stabilité Hölderienne pour un problème inverse de sources ponctuelles

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ABSTRACT

In this Note we establish a Hölder stability estimate for an inverse pointwise source elliptic problem.

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RÉSUMÉ

Nous établissons dans cette Note un résultat de stabilité Hölderienne dans un problème inverse de sources ponctuelles.

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Soit Ω un domaine borné de \mathbb{R}^3 de frontière Γ supposée suffisamment régulière. On considère dans cette Note le problème de détermination du terme source F dans le problème elliptique

$$\Delta u + k^2 u = F \quad \text{dans } \Omega,$$

à partir des données de Cauchy $(f, g) := (u|_{\Gamma}, \frac{\partial u}{\partial n}|_{\Gamma})$. Ici k est un nombre réel donné et la source F est supposée être combinaison linéaire finie de masses de Dirac $F = \sum_{j=1}^m \lambda_j \delta_{S_j}$, où $\lambda_j \neq 0$ et les points S_j sont supposés mutuellement distincts.

L'objet de cette Note est d'établir un résultat de stabilité Hölderienne des localités S_j , m étant supposé connu. Pour les questions de l'unicité et de l'identification, on renvoie, par exemple, à [5]. Avant de présenter notre principal résultat, nous commençons par introduire quelques notations et donnons quelques définitions.

Soient S un point de Ω et $d(\Gamma, S)$ la distance Euclidienne entre le bord Γ et S . Nous posons $\alpha = \min_{1 \leq j \leq m} d(\Gamma, S_j)$, définissons $\Omega_\alpha = \{S \in \Omega : d(\Gamma, S) \geq \alpha\}$ et posons $\beta = \text{diam}(\Omega) - \alpha$ où $\text{diam}(\Omega)$ désigne le diamètre de Ω .

Soient $S_j = (S_{j,1}, S_{j,2}, S_{j,3})$ un point de Ω , et $P_j \equiv S_{j,1} + iS_{j,2}$ sa projection sur le plan complexe xy . Comme dans la pratique les points sources sont déterminés par leurs projections sur les plans xy et yz (voir par exemple [5]), il est naturel d'étudier la stabilité des projections que nous supposons mutuellement distincts. Au moyen du théorème de Hall–Rado [10], nous obtenons le résultat de stabilité suivant :

Théorème 0.1 (Stabilité des locations S_j). Soit u^ℓ , pour $\ell = 1, 2$ la solution de (1) correspondant à la source $F^\ell = \sum_{j=1}^m \lambda_j^\ell \delta_{S_j^\ell}$ caractérisée par la configuration $(\lambda_j^\ell, S_j^\ell)_{1 \leq j \leq m}$. Soient $(P_j^\ell)_{1 \leq j \leq m}$ les projections des points sources S_j^ℓ sur le plan complexe xy .

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Supposant que $(S_j^\ell)_{1 \leq j \leq m} \subset \Omega_\alpha$ et notant $(f^\ell, g^\ell) := (u|_\Gamma, \frac{\partial u}{\partial n}|_\Gamma)$, pour $\ell = 1, 2$, alors, il existe une permutation π des entiers naturels $1, \dots, m$, telle que

$$\max_{1 \leq j \leq m} \|P_j^2 - P_{\pi(j)}^1\| \leq \left[\frac{\sqrt{|\Gamma|} \beta^{2m-1}}{c_1 \varrho^{m-1}} [\|g^2 - g^1\|_{L^2(\Gamma)} + c_2 \|f^2 - f^1\|_{L^2(\Gamma)}] \right]^{\frac{1}{m}}$$

où $c_1 = \min_{1 \leq j \leq m} (|\lambda_j^1|, |\lambda_j^2|)$, $c_2 = \sqrt{2 \frac{(2m-1)^2}{\beta^2} + k^2}$, $\varrho = \min_{1 \leq j, \kappa \leq m, j \neq \kappa, 1 \leq \ell \leq 2} \|P_j^\ell - P_\kappa^\ell\|$ et $|\Gamma| = \int_\Gamma ds$.

1. Introduction and main result

Inverse source problems (IPs) are very important in science, engineering and bioengineering. Among these, inverse source problems have attracted great attention of many researchers over recent years because of their applications to many practical examples. We quote two of these applications for which there is abundant literature: identification of pollution sources in the environment (see for example [3,8], and references therein) and current dipolar sources in the so-called inverse electroencephalography (EEG) and magnetoencephalography (MEG) problems, see for example [6,7].

In this Note, we consider the problem of determining a source F in the following elliptic equation

$$\Delta u + k^2 u = F \quad \text{in } \Omega, \tag{1}$$

from the Cauchy data $(f, g) := (u|_\Gamma, \frac{\partial u}{\partial n}|_\Gamma)$ prescribed on the boundary Γ of Ω , where $\Omega \subset \mathbb{R}^3$ is an open bounded domain with a sufficiently regular boundary Γ . Here k is a fixed real number assumed to be known.

One of difficulties of the inverse source problem from boundary measurements concerns the non-uniqueness of the source, for example because of the possible existence of non-radiating sources. Also, it is obvious that, if we add to the solution u of (1) any function or distribution v with support in Ω , we get a solution of the same equation with a (possibly) different RHS source F and the same boundary data. Thus, in the general case, a source F cannot be identified from boundary measurements when no a priori information is available.

In this Note we assume that the sources F are a finite linear combination of point sources given by

$$F = \sum_{j=1}^m \lambda_j \delta_{S_j}, \quad S_j \in \Omega, \lambda_j \neq 0 \tag{2}$$

where δ_S stands for the Dirac distribution at point S , m is a positive integer, for $j = 1, \dots, m$, S_j are points in Ω assumed to be mutually distinct and $\lambda_j \neq 0$ are scalar quantities.

Several important questions arise concerning this inverse problem. First, is the source F , precisely m , S_j , λ_j , uniquely determined? Second, are there effective algorithms to construct the source F ? Third, does F stably depend on the Cauchy data (f, g) ?

Uniqueness and identification has been completely solved (see for instance [2,5]). For stability from boundary measurements, only some partial conditional stability results exist. Let us mention that, in Cannon et al. [1] the authors have considered the 2D case of the problem of locating dense masses in the earth from gravimetry data taken at the surface or in the air. They obtained a logarithm type stability estimate, assuming that, the poles are well separated and their respective strengths (or residue) are large enough. The same problem in 3D case was studied by Vessella in [11]. In this work, the author has obtained a conditional Hölder type stability estimate, taking the same assumptions considered in [1]. Let us mention a conditional Lipschitz stability estimate obtained in the work of El Badia [4], with a constant that increases according to the number of sources. Let us also mention that, Kang et al. in [9] have obtained, for monopolar sources considered in a disc, a similar estimate to that shown in [4]. Their result was derived from algebraic relations similar to those obtained in El Badia et al. [2].

Before formulating the main results of this paper, we introduce some notation and specify additional information. First, let S be a point of Ω and $d(\Gamma, S)$ be the Euclidean distance between the boundary Γ and S . We pose $\alpha = \min_{1 \leq j \leq m} d(\Gamma, S_j)$ which is greater than zero since $S_j \in \Omega$, define the set

$$\Omega_\alpha = \{S \in \Omega : d(\Gamma, S) \geq \alpha\}$$

and pose $\beta = \text{diam}(\Omega) - \alpha$, where $\text{diam}(\Omega)$ denotes the diameter of Ω .

Let $S_j = (S_{j,1}, S_{j,2}, S_{j,3})$ and $P_j \equiv S_{j,1} + iS_{j,2}$ its projection onto the xy -complex plane. Since in practice, localization of point sources is done by determining their projections on the xy - and yz -planes (see for example [5]), it is natural to study the stability of the projected points. However, we need to know if the projection P_j are mutually distinct, which is necessary to use the algebraic method that is behind our method for establishing stability. Indeed, one can remark that there are only a finite number of planes containing the origin and upon which at least two points among (S_j) are projected onto the same point on these planes. So, if a basis is chosen randomly, one is almost sure that the S_j are projected onto distinct points on every coordinate plane. Therefore, in the following, the projected points P_j will be assumed to be distinct.

Considering a set of the projected sources $\mathbf{P} = (P_j)_{1 \leq j \leq m}$, we introduce the following real coefficient

$$\varrho = \min_{1 \leq j, \ell \leq m, j \neq \ell} \|P_j - P_\ell\| \tag{3}$$

which will henceforth be called “separability coefficient” of the projected sources \mathbf{P} .

Let now $\mathbf{S}^\ell = (S_j^\ell)_{1 \leq j \leq m}$, $\ell = 1, 2$ be two configurations such that

$$\mathbf{S}^\ell \subset \Omega_\alpha, \quad \text{for } \ell = 1, 2. \tag{4}$$

We can now precisely our main result.

Theorem 1.1 (Stability of locations S_j). *Let u^ℓ , for $\ell = 1, 2$ be the solution of (1) corresponding to the sources $F^\ell = \sum_{j=1}^m \lambda_j^\ell \delta_{S_j^\ell}$ characterized by the configurations $(\lambda_j^\ell, S_j^\ell)_{1 \leq j \leq m}$. Let $\mathbf{P}^\ell = (P_j^\ell)_{1 \leq j \leq m}$ be the corresponding projected point sources on the xy -plane. Assuming (4) and noting $(f^\ell, g^\ell) := (u_{|r}^\ell, \frac{\partial u^\ell}{\partial n}|_r)$ for $\ell = 1, 2$, then, there exists a permutation π of the integer $1, \dots, m$, such that the following estimate holds*

$$\max_{1 \leq j \leq m} \|P_j^2 - P_{\pi(j)}^1\| \leq \left[\frac{\sqrt{|\Gamma|} \beta^{2m-1}}{c_1 \varrho^{m-1}} [\|g^2 - g^1\|_{L^2(\Gamma)} + c_2 \|f^2 - f^1\|_{L^2(\Gamma)}] \right]^{\frac{1}{m}} \tag{5}$$

where $c_1 = \min_{1 \leq j \leq m} (|\lambda_j^1|, |\lambda_j^2|)$, $c_2 = \sqrt{2 \frac{(2m-1)^2}{\beta^2} + k^2}$, $|\Gamma| = \int_\Gamma ds$ and ϱ is the separability coefficient of \mathbf{P}^1 and \mathbf{P}^2 .

2. Proof of stability estimate

Before establishing our stability estimate, we need to recall the definition of the Hausdorff distance between the configurations \mathbf{P}^1 and \mathbf{P}^2

$$d_H(\mathbf{P}^1, \mathbf{P}^2) = \max \left[\max_{1 \leq \ell \leq m} \min_{1 \leq j \leq m} \|P_\ell^2 - P_j^1\|, \max_{1 \leq \ell \leq m} \min_{1 \leq j \leq m} \|P_\ell^1 - P_j^2\| \right]$$

and we also need the following lemma which consists of estimating the Hausdorff distance between the two configurations \mathbf{P}^1 and \mathbf{P}^2 .

Lemma 2.1. *Let u^ℓ , for $\ell = 1, 2$ be the solution of (1) corresponding to the sources $F^\ell = \sum_{j=1}^m \lambda_j^\ell \delta_{S_j^\ell}$ characterized by the configurations $(\lambda_j^\ell, S_j^\ell)_{1 \leq j \leq m}$. Let $(P_j^\ell)_{1 \leq j \leq m}$ be the corresponding projected point sources on the xy -plane. Assuming (4) and noting $(f^\ell, g^\ell) := (u_{|r}^\ell, \frac{\partial u^\ell}{\partial n}|_r)$ for $\ell = 1, 2$, then*

$$d_H(\mathbf{P}^1, \mathbf{P}^2) \leq \left[\frac{\sqrt{|\Gamma|} \beta^{2m-1}}{c_1 \varrho^{m-1}} [\|g^2 - g^1\|_{L^2(\Gamma)} + c_2 \|f^2 - f^1\|_{L^2(\Gamma)}] \right]^{\frac{1}{m}}$$

where $c_1, c_2, |\Gamma|$ and ϱ are defined in Theorem 1.1.

Proof. First of all, we introduce the following functional space:

$$\mathcal{H}_k = \{v \in H^1(\Omega): \Delta v + k^2 v = 0\}$$

and define the following operator

$$\mathcal{R}(v, f, g) = \int_\Gamma \left(gv - f \frac{\partial v}{\partial n} \right) ds.$$

Noting that multiplying Eq. (1) by v , element of \mathcal{H}_k , integrating by parts and using Green formula led to the following

$$\sum_{j=1}^m \lambda_j v(S_j) = \mathcal{R}(v, f, g) \tag{6}$$

when F is given by (2).

Thus, the question is now to choose functions v in \mathcal{H}_k allowing us to establish the desired stability results. Indeed, for $\ell = 1, \dots, m$ we consider the following functions

$$\Phi_\ell(x, y) = \prod_{j=1}^m (x + iy - P_j^1) \prod_{j \neq \ell}^m (x + iy - P_j^2)$$

and $\Psi_\ell(x, y, z) = \Phi_\ell(x, y)e^{-ikz}$.

First, observe that Φ_ℓ is a harmonic function and then the function Ψ_ℓ belongs to the space \mathcal{H}_k . Therefore, from (6), we obtain

$$\sum_{j=1}^m \lambda_j^2 \Phi_\ell(P_j^2) e^{-ikS_{j,3}^2} = R(\Psi_\ell, f^2, g^2), \quad \text{and} \quad \sum_{j=1}^m \lambda_j^1 \Phi_\ell(P_j^1) e^{-ikS_{j,3}^1} = R(\Psi_\ell, f^1, g^1).$$

Taking the difference between the previous two sums, for all $\ell = 1, \dots, m$, we get

$$\lambda_\ell^2 \Phi_\ell(P_\ell^2) e^{-ikS_{\ell,3}^2} = R(\Psi_\ell, f^2 - f^1, g^2 - g^1). \tag{7}$$

Then, using the definitions of R and Φ_ℓ , the formula (7) can be rewritten as follows

$$\lambda_\ell^2 e^{-ikS_{\ell,3}^2} \prod_{j=1}^m (P_\ell^2 - P_j^1) \prod_{j \neq \ell}^m (P_\ell^2 - P_j^2) = \int_\Gamma \left[(g^2 - g^1) \Psi_\ell - (f^2 - f^1) \frac{\partial \Psi_\ell}{\partial n} \right] ds.$$

Moreover, using the *separability coefficient* (3) and thanks to the smoothness of the function Ψ_ℓ , we get, by Hölder estimate, the following estimate:

$$c_1 \left(\min_{1 \leq j \leq m} \|P_\ell^2 - P_j^1\| \right)^m \leq \frac{1}{Q^{m-1}} \left[\|\Psi_\ell\|_{L^2(\Gamma)} \|g^2 - g^1\|_{L^2(\Gamma)} + \left\| \frac{\partial \Psi_\ell}{\partial n} \right\|_{L^2(\Gamma)} \|f^2 - f^1\|_{L^2(\Gamma)} \right].$$

Now, one has to estimate

$$\|\Psi_\ell\|_{L^2(\Gamma)} \quad \text{and} \quad \left\| \frac{\partial \Psi_\ell}{\partial n} \right\|_{L^2(\Gamma)}.$$

Indeed, according to the definition of β and using the Cauchy–Schwarz inequality, one obtains

$$\|\Psi_\ell\|_{L^2(\Gamma)} \leq \sqrt{|\Gamma|} \beta^{2m-1} \quad \text{and} \quad \left\| \frac{\partial \Psi_\ell}{\partial n} \right\|_{L^2(\Gamma)} \leq \sqrt{|\Gamma|} \left[2 \frac{(2m-1)^2}{\beta^2} + k^2 \right]^{\frac{1}{2}} \beta^{2m-1}$$

where $|\Gamma| = \int_\Gamma ds$. Thus,

$$\max_{1 \leq \ell \leq m} \min_{1 \leq j \leq m} \|P_\ell^2 - P_j^1\| \leq \left[\frac{\sqrt{|\Gamma|} \beta^{2m-1}}{c_1 Q^{m-1}} [\|g^2 - g^1\|_{L^2(\Gamma)} + c_2 \|f^2 - f^1\|_{L^2(\Gamma)}] \right]^{\frac{1}{m}}. \tag{8}$$

We repeat the same procedure replacing the function Φ_ℓ by $\tilde{\Phi}_\ell$, for $\ell = 1, \dots, m$

$$\tilde{\Phi}_\ell(x, y) = \prod_{j=1}^m (x + iy - P_j^2) \prod_{j \neq \ell}^m (x + iy - P_j^1)$$

we can prove similarly to (8) that

$$\max_{1 \leq \ell \leq m} \min_{1 \leq j \leq m} \|P_\ell^1 - P_j^2\| \leq \left[\frac{\sqrt{|\Gamma|} \beta^{2m-1}}{c_1 Q^{m-1}} [\|g^2 - g^1\|_{L^2(\Gamma)} + c_2 \|f^2 - f^1\|_{L^2(\Gamma)}] \right]^{\frac{1}{m}}. \tag{9}$$

Finally, taking the maximum between (8) and (9), we get the desired result. \square

In order to achieve the proof of our stability estimate on point sources, we need to recall the following theorem, borrowed from graph theory and called Hall–Rado Theorem (see for instance Rado [10]).

Theorem 2.2 (Hall–Rado). Consider an even graph having $2m$ points a_1, \dots, a_m and b_1, \dots, b_m . We connect some pairs (a_i, b_j) such that for every $k \in \{1, \dots, m\}$ and every subsequence $(a_{j_1}, \dots, a_{j_k})$ of (a_1, \dots, a_m) , at least k elements b_j of the sequence (b_1, \dots, b_m) are connected to one of them. Then there exists a permutation π of the integer $1, \dots, m$ such that a_j is connected to $b_{\pi(j)}$ for every j .

Now, from Hall–Rado Theorem and Lemma 2.1, we can establish stability of locations.

Proof of Theorem 1.1. Collecting Lemma 2.1 and Theorem 2.2, we can easily prove that the stability estimate (5) holds. Indeed, considering the even graph $(\mathbf{P}^1, \mathbf{P}^2)$ with points $P_\ell^1, P_\ell^2, \ell = 1, \dots, m$ and noting that P_ℓ^1 is connected to P_j^2 if $\|P_\ell^1 - P_j^2\| \leq d_H(\mathbf{P}^1, \mathbf{P}^2)$. Now, according to the definition of the Hausdorff distance $d_H(\mathbf{P}^1, \mathbf{P}^2)$, it is easy to see that, for every $\ell \in \{1, \dots, m\}$ and every subsequence $(P_{j_1}^2, \dots, P_{j_\ell}^2)$ of \mathbf{P}^2 , at least ℓ elements P_j^1 of the sequence \mathbf{P}^1 are connected to one of them. Then the desired result is obtained from Hall–Rado Theorem and Lemma 2.1. \square

3. Remark and conclusion

1. When $k \neq 0$, our method is valid only in \mathbb{R}^3 , while it is in \mathbb{R}^2 and \mathbb{R}^3 for $k = 0$.
2. Stability results for the intensities and dipole sources are in progress in a future work.

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