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## Ordinary Differential Equations

# Lyapunov exponent of a stochastic SIRS model

Exposant de Lyapunov d'un modèle SIRS stochastique

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#### ABSTRACT

We consider a SIRS (susceptible–infected–removed–susceptible) model influenced by random perturbations. We prove that the solutions are positive for positive initial conditions and are global, that is, there is no finite explosion time. We present necessary and sufficient conditions for the almost sure asymptotic stability of the steady state of the stochastic system.

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## RÉSUMÉ

Nous considérons un modèle de type SIRS avec perturbation stochastique. Nous démontrons que les solutions sont positives pour des conditions initiales positives et sont définies globalement. Nous présentons des conditions nécessaires et suffisantes pour la stabilité asymptotique presque sûre de la solution triviale du système stochastique.

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### 1. Introduction

In the understanding of different scenarios for disease transmissions and behavior of epidemics, many models in the literature represent dynamics of diseases by systems of ordinary differential equations. The dynamic behaviors of the SIRS models have been investigated by several authors. In the 1920s, a Kermack–Mackendrick epidemic SIRS (susceptible–infected–removed–susceptible) model [5] was proposed. Since then, many people have studied the SIRS disease model (acquired immunity is permanent or acquired immunity is temporary) with different variations in its incidence rate, at which susceptibles become infectives, see [7,8,13].

The deterministic SIRS model exhibiting loss of immunity is the following:

$$S'(t) = -\beta S(t)I(t) - \mu S(t) + \gamma R(t) + \mu,$$
  

$$I'(t) = \beta S(t)I(t) - (\lambda + \mu)I(t),$$
  

$$R'(t) = \lambda I(t) - (\mu + \gamma)R(t),$$

where S(t) is the number of members of the population susceptible to the disease, I(t) is the number of infective members and R(t) is the number of members who have been removed from the possibility of infection through full immunity. The

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population considered has a constant size N which is normalized to 1, that is S(t) + I(t) + R(t) = 1 for all  $t \ge 0$ . This equality is preserved for the stochastic perturbed system in the sequel. We refer to [7] for details about this model.

Parameters in the system are as follows:  $\mu$  represents the birth and death rate, the constant  $\lambda$  represents the recovery rate of infected people,  $\beta$  is the transmission rate, and  $\gamma$  is the per capita rate of loss of immunity. Of course,  $\mu > 0$ ,  $\lambda > 0$ ,  $\beta > 0$ . We shall assume that  $\gamma \ge 0$ , the case where  $\gamma = 0$  corresponds to the SIR model. It is easy to see that the above system always has a disease-free equilibrium (*i.e.* boundary equilibrium)  $E_0 = (1, 0, 0)$ . One of the main issues in the study of the behavior of epidemics is the analysis of the steady states of the model and their stabilities.

A stochastic version of the present model (with or without delay) has been considered in [9] for the stability of the disease-free equilibrium, where white noises appear only in the first two equations. The stability of the disease-free of a SIR model has been studied in [12]. The stability of the endemic equilibrium for the SIR model has been studied in [3]. The method used in these papers is that of stochastic Lyapunov functions. Sufficient conditions are given for the stabilities of the corresponding equilibria. Most of the studies in the literature concerning stabilities provide only sufficient conditions. One of the aims of the present Note is to give necessary and sufficient conditions for the stability of a stochastic perturbation of the above model.

The stochastic model considered in [9,12] is the following:

$$dS(t) = \left[-\beta S(t)I(t) - \mu S(t) + \gamma R(t) + \mu\right] dt - \sigma S(t)I(t) dw(t),$$
  

$$dI(t) = \left[\beta S(t)I(t) - (\lambda + \mu)I(t)\right] dt + \sigma S(t)I(t) dw(t),$$
  

$$dR(t) = \left[\lambda I(t) - (\mu + \gamma)R(t)\right] dt,$$

where w(t) is a one-dimensional standard Brownian motion defined on a complete probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, P)$ . One notices that there is no white noise perturbation in the third equation.

For the case of stochastic SIR model, that is when  $\gamma = 0$ , sufficient conditions  $(0 < \beta < \min\{\lambda + \mu - \frac{\sigma^2}{2}, 2\mu\})$  are given in [12] for the stability of the disease-free equilibrium. In [9], the stability is proved under the condition  $0 < \beta < \lambda + \mu - \frac{\sigma^2}{2}$ for the stochastic SIRS model. Numerical simulations are also given in [9,12] to support the conjecture that the equilibrium of the model is still stable under the more general condition  $0 < \beta < \lambda + \mu + \frac{\sigma^2}{2}$ . In the present Note we prove this conjecture for systems with more general type of noise perturbations, namely systems

of the form

$$dS(t) = \left[-\beta S(t)I(t) - \mu S(t) + \gamma R(t) + \mu\right] dt - \left[\sigma_1 S(t)I(t) + \sigma_2 S(t)R(t)\right] dw(t),$$
  

$$dI(t) = \left[\beta S(t)I(t) - (\lambda + \mu)I(t)\right] dt + \sigma_1 S(t)I(t) dw(t),$$
  

$$dR(t) = \left[\lambda I(t) - (\mu + \gamma)R(t)\right] dt + \sigma_2 S(t)R(t) dw(t),$$
  
(1)

where w(t) is as above, and  $\sigma_1, \sigma_2 \in \mathbf{R}$ . When  $\sigma_2 = 0$  we recover the system studied in [9,12]. The white noises of this system depend on two parameters while in the preceding one they depend only on one parameter.

To study the stochastic stabilities of the above system, several definitions of stochastic stabilities could be used, where the convergence of a stochastic sequence could be interpreted in different ways [2,4,6,10]. One of the most important definitions of stochastic stabilities is the almost sure asymptotic stability, for which necessary and sufficient conditions are always determined by the largest Lyapunov exponent [2]. To the best of the authors' knowledge, there is no result in the study of the almost sure asymptotic stability of the SIRS model.

When dealing with a model for population, it is necessary to prove that the solutions are positive for any positive initial conditions, and are defined globally for all  $t \ge 0$ , *i.e.* there is no finite explosion time. This question is not handled in the papers cited above. One of the aims of this Note is to give a positive answer for it.

## 2. Global and positive solutions, almost sure asymptotic stability

Concerning the existence of global and positive solutions, we obtain the following result:

**Theorem 1.** Let the initial data  $(\xi_1, \xi_2, \xi_3)$  be an  $\mathbb{R}^3$ -valued  $\mathcal{F}_0$ -measurable random variable satisfying  $\mathbb{E}[\xi_1^2 + \xi_2^2 + \xi_3^2] < \infty$  and  $\xi_1 > 0, \ \xi_2 > 0, \ \xi_3 > 0$  a.s. Then there is a unique solution  $(S(t, \omega), I(t, \omega), R(t, \omega))$  to system (1), defined for all  $t \ge 0$ , verifying the initial conditions  $S(0, \omega) = \xi_1$ ,  $I(0, \omega) = \xi_2$ ,  $R(0, \omega) = \xi_3$ , and the solution is positive for all  $t \ge 0$  with probability 1, namely  $S(t, \omega) > 0$ ,  $I(t, \omega) > 0$  and  $R(t, \omega) > 0$  for all  $t \ge 0$  almost surely.

According to the Oseledec multiplicative ergodic theorem [11,1], the necessary and sufficient conditions for the almost sure asymptotic stability, of the trivial solution of the system, is that the largest Lyapunov exponent of the linearized system is negative.

To compute the largest Lyapunov exponent of the linearized system, we make the changes of variables  $u_1 = S - 1$ ,  $u_2 = I$ ,  $u_3 = R$ , so that the origin will represent the disease-free equilibrium. Then we consider the linearized system of (1), which is the following:

$$du_{1}(t) = \left[-\beta u_{2}(t) - \mu u_{1}(t) + \gamma u_{3}(t)\right] dt - \left(\sigma_{1} u_{2}(t) + \sigma_{2} u_{3}(t)\right) dw(t),$$
  

$$du_{2}(t) = \left(\beta - \lambda - \mu\right) u_{2}(t) dt + \sigma_{1} u_{2}(t) dw(t),$$
  

$$du_{3}(t) = \left[\lambda u_{2}(t) - (\mu + \gamma) u_{3}(t)\right] dt + \sigma_{2} u_{3}(t) dw(t).$$
(2)

We denote by  $u(t, u_0)$  the unique solution of system (2) verifying the initial condition  $u(0) = u_0 = (u_{10}, u_{20}, u_{30})$ . We study the largest Lyapunov exponent of system (2), more precisely, the quantity  $\limsup_{t \to +\infty} \frac{1}{t} \log ||u(t, u_0)||$ . The second equation in (2) is a scalar linear stochastic differential equation. So one can solve it explicitly. We obtain

The second equation in (2) is a scalar linear stochastic differential equation. So one can solve it explicitly. We obtain  $u_2(t, u_0) = e^{(\beta - \lambda - \mu - \frac{\sigma_1^2}{2})t + \sigma_1 w(t)} u_{20}$ . It is clear that

 $u_{2}(t, u_{0}) = e^{\alpha}$   $r = 2^{\alpha}$   $u_{20}$ . It is clear that  $\sigma^{2}$ 

$$\limsup_{t\to+\infty}\frac{1}{t}\log|u_2(t,u_0)|=\beta-\lambda-\mu-\frac{\sigma_1^2}{2}.$$

The next step is to give estimate for  $u_1(t, u_0)$  and  $u_3(t, u_0)$ . One can prove the following, for  $t \ge e$ :

$$\begin{aligned} \left| u_1(t, u_0) \right| &\leq C_1 e^{-\mu t} + C_2 e^{-a't + \sigma_2 w(t)} + C_3 e^{-at + \sigma_1 w(t)} + C_4 \left[ t e^{-\mu t} + e^{-at} + e^{-\mu t} \right] e^{|\sigma_1| c \sqrt{2t \log \log t}} \\ &+ C_5 \left[ t e^{-a't + \sigma_2 w(t)} + e^{-at + \sigma_2 w(t)} + e^{-a't + \sigma_2 w(t)} \right] e^{|\sigma_1 - \sigma_2| c \sqrt{2t \log \log t}}, \end{aligned}$$

where the  $C_j$  are positive constants, and  $a' = \mu + \gamma + \frac{1}{2}\sigma_2^2$ ,  $a = \lambda + \mu + \frac{\sigma_1^2}{2} - \beta$ . Therefore

$$\limsup_{t\to+\infty}\frac{1}{t}\log|u_1(t)|=\max\{-\mu,-a',-a\}.$$

The assertion for  $u_3(t, u_0)$  can be proved similarly. Therefore we have the following result:

## Theorem 2.

(i) The largest Lyapunov exponent of the linearized system at (1, 0, 0) of system (1) is equal to  $\max\{\beta - (\lambda + \mu + \frac{\sigma_1^2}{2}), -\gamma - \mu - \frac{\sigma_2^2}{2}, -\mu\}$ .

(ii) The trivial solution (1, 0, 0) of system (1) is almost surely asymptotically stable if and only if  $\beta < \lambda + \mu + \frac{\sigma_1^2}{2}$ .

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