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Partial Differential Equations

A Carleman estimate for the two dimensional heat equation with mixed boundary conditions

Inégalité de Carleman pour l'équation de la chaleur avec conditions mixtes en dimension deux

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A B S T R A C T

It is well known that in a regular domain, the solutions of the Laplace equation with mixed boundary conditions can present a singular part. In this work, we prove a Carleman estimate for the two dimensional domain heat equation in presence of these singularities.

R É S U M É

Il est bien connu que dans un ouvert régulier, les solutions du problème mêlé pour l'équation de Laplace présentent des singularités. Le but de ce travail est d'établir une inégalité de Carleman pour l'équation de la chaleur en dimension deux en présence de ces singularités.

1. Introduction

Let Ω be a bounded open connected set of $\mathbb{R}^2$ with C^2 boundary $\Gamma = \partial\Omega$. Let Γ_D and Γ_N be two subsets of Γ such that: $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \{S_1, S_2\}$. Let $\omega \subset \Omega$ be a non-empty open subset. For $T > 0$, we set $Q_T = \Omega \times (0,T)$, $\Sigma_{DT} = \Gamma_D \times (0,T)$, $\Sigma_{NT} = \Gamma_N \times (0,T)$ and $\Sigma_T = \Gamma \times (0,T)$. We will denote by $\nu(x)$ the outward unit normal to Ω at $x \in \Gamma$. We consider the following mixed boundary value problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_N, \end{cases} \tag{1}$$

where f is given in $L^2(\Omega)$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative of u. It is well known, see [8], that the solution of (1) is not in $H^2(\Omega)$, and more precisely the solution, according to [7] is given by:

$$u(r,\theta) = u_R(r,\theta) + C_1\sqrt{r_1}\sin\frac{\theta_1}{2}\chi_1 + C_2\sqrt{r_2}\cos\frac{\theta_2}{2}\chi_2, \tag{2}$$

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where $u_R \in H^2(\Omega)$ is the regular part, (r_j, θ_j) are the local polar coordinates at S_j, C_j are real constants, χ_j are cut-off functions such that $0 \leqslant \chi_j \leqslant 1$ and $\chi_j = 1$ on a neighborhood of S_j.

The aim of this note is to establish a Carleman estimate for the following problem:

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_{DT}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma_{NT}, \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{3}$$

where $u_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$.

Carleman estimates have many applications varying from the quantification of the unique continuation problems, inverse problems to stabilization and control theory. These applications are the motivation to prove a suitable Carleman estimate for our problem. To the best of our knowledge, very few results on Carleman estimates in the presence of singularities have been established. We cite [2] for the Laplace equation for a domain with a corner, [1] for the heat equation in a singular domain and [3] for the wave equation with mixed conditions using microlocal approach. Our methodology here is in a similar spirit of [1,4–6].

In order to get well-possedness for (3), we define the following spaces:

$$V = \{u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma_D\}$$

and

$$D(-\Delta) = \left\{u \in V;\ \Delta u \in L^2(\Omega);\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_N\right\},$$
$$= \left\{u \in V \cap H^2(\Omega);\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_N\right\} \oplus \operatorname{span}\left\{r^{\frac{1}{2}} \sin\frac{\theta}{2}, r^{\frac{1}{2}} \cos\frac{\theta}{2}\right\}.$$

Problem (3) has a unique solution $u \in C([0,T]; L^2(\Omega)) \cap C((0,T); D(-\Delta)) \cap C^1((0,T); L^2(\Omega))$. Note that, even for very smooth data f and u_0, the solution of (3) is not regular near S_1 and S_2.

2. Main result

In the following, for $k = 0, 1$, we set:

$$\xi_k(x,t) = \frac{e^{(-1)^k \lambda \beta(x)}}{t(T-t)}, \qquad \alpha_k(x,t) = \frac{e^{2\lambda \|\beta\|_\infty} - e^{(-1)^k \lambda \beta(x)}}{t(T-t)}. \tag{4}$$

Here, $\lambda \geqslant 1$ is a parameter and $\beta = \beta(x)$ is a function satisfying:

$$\beta \in C^2(\overline{\Omega}), \qquad \beta(x) > 0 \quad \text{in } \Omega, \qquad \beta(x) = 0 \quad \text{on } \partial\Omega, \qquad |\nabla \beta| > 0 \quad \text{on } \overline{\Omega \setminus \omega'}, \tag{5}$$

where $\omega' \Subset \omega$ is a non-empty open set. The existence of β satisfying (5) is proved in [6].

We state our main result, for $\alpha = \alpha_0$ and $\xi = \xi_0$:

Theorem 2.1. *Given $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$. There exist s_0, λ_0 and $C = C(\Omega, \omega)$ such that for any $s > s_0$, $\lambda > \lambda_0$ the solution of (3) satisfies*:

$$I(u, \xi_0, \alpha_0, Q_T) \leqslant C\left(\int_{Q_T} e^{-2s\alpha_0} |f|^2 \,dx\,dt + s^3\lambda^4 \int_{\omega \times (0,T)} \xi_0^3 e^{-2s\alpha_0} |u|^2 \,dx\,dt\right), \tag{6}$$

where

$$I(u, \xi, \alpha, Q_T) = \int_{Q_T} e^{-2s\alpha}\left((s\xi)^{-1}\left(|\partial_t u|^2 + |\Delta u|^2\right) + s\lambda^2 \xi |\nabla u|^2 + s^3 \lambda^4 \xi^3 |u|^2\right) dx\,dt. \tag{7}$$

Sketch of the proof. The proof will be given in four steps.

Regularization of the solution: The regularity of the solution u is not sufficient to do some integrations by parts. We then approximate u by a sequence of regular functions (u_n). Spaces $D(-\Delta)$ and $D((-\Delta)^2)$ are dense in $L^2(\Omega)$, then for $u_0 \in L^2(\Omega)$ and $f \in L^2(0,T; L^2(\Omega))$, there exist sequences $(u_n^0)_n \subset D((-\Delta)^2)$ and $(f_n)_n \subset C^1(0,T; D(-\Delta))$ such that $(u_n^0)_n$ converges to u_0 in $L^2(\Omega)$ and $(f_n)_n$ converges to f in $L^2(0,T; L^2(\Omega))$, then the problem (3) with $f = f_n$ and $u_0 = u_n^0$ has a unique solution $u_n \in C^2((0,T); L^2(\Omega)) \cap C^1([0,T]; D(-\Delta))$ and we can prove the following lemma:

Lemma 2.1. *For $k = 0, 1$, set $\psi_{n,k}(x,t) = e^{-s\alpha_k(x,t)} u_n(x,t)$ and $\psi_k(x,t) = e^{-s\alpha_k(x,t)} u(x,t)$, we have*

1. *$(u_n)_n$ converges to u in $L^2(0,T;V)$,*
2. *$(\Delta\psi_{n,k})_n$ converges to $(\Delta\psi_k)$ in $L^2(Q_T)$.*

In the sequel, and for simplicity, we will drop the index n.

Approximation of the domain: To remedy the lack of regularity of the solution near S_1 and S_2, we set, for $\varepsilon > 0$

$$\Omega_\epsilon = \Omega \setminus \bigcup_{k=1}^{2} B(S_k,\varepsilon), \qquad \partial\Omega_\varepsilon = \Gamma_D^\varepsilon \cup \Gamma_N^\varepsilon \cup C_\varepsilon^1 \cup C_\varepsilon^2,$$

$$Q_{\varepsilon,T} = \Omega_\varepsilon \times (0,T), \qquad \Sigma_{\varepsilon,T} = \partial\Omega_\varepsilon \times (0,T), \qquad C_\varepsilon^k = \partial B(S_k,\varepsilon) \cap \Omega,$$

where $B(S_k,\varepsilon)$ is the ball of radius ε and centred in S_k.

Derivation of the Carleman estimate: For $k = 0, 1$, let:

$$\xi_k(x,t) = \frac{e^{(-1)^k\lambda\beta(x)}}{t(T-t)}, \qquad \alpha_k(x,t) = \frac{e^{2\lambda\|\beta\|_\infty} - e^{(-1)^k\lambda\beta(x)}}{t(T-t)}, \tag{8}$$

we set:

$$\psi_k(x,t) = e^{-s\alpha_k(x,t)} u(x,t) \tag{9}$$

and

$$L\psi_k = M_1\psi_k + M_2\psi_k = F_k,$$

where

$$\begin{cases} M_1\psi_k = 2s\lambda^2\xi_k|\nabla\beta|^2\psi_k + 2(-1)^k s\lambda\xi_k\nabla\beta.\nabla\psi_k + \partial_t\psi_k, \\ M_2\psi_k = -s^2\lambda^2|\nabla\beta|^2\xi_k^2\psi_k - \Delta\psi_k + s\partial_t\alpha_k\psi_k, \\ F_k = e^{-s\alpha_k} f - (-1)^k s\lambda\xi_k\Delta\beta\psi_k + s\lambda^2\xi_k|\nabla\beta|^2\psi_k, \end{cases}$$

$(M_1\psi_k)_i$, $(M_2\psi_k)_j$ are respectively the i-th and the j-th term of $M_1\psi_k$ and of $M_2\psi_k$.

$$\|M_1\psi_k\|^2_{L^2(Q_T)} + \|M_2\psi_k\|^2_{L^2(Q_T)} + 2\sum_{i,j=1}^{3} \langle (M_1\psi_k)_i, (M_2\psi_k)_j \rangle_{L^2(Q_T)} = \|F_k\|^2_{L^2(Q_T)}. \tag{10}$$

Using integration by parts in (10), we derive the following inequality:

Lemma 2.2. *There exist s_0, λ_0 and $C = C(\Omega,\omega)$ such that for any $s > s_0$, $\lambda > \lambda_0$,*

$$I(\psi_k,\xi_k,\alpha_k,Q_{\varepsilon,T}) + J(\psi_k,\xi_k,\alpha_k,\Sigma_{\varepsilon,T}) \leqslant C\left(\int_{Q_{\varepsilon,T}} e^{-2s\alpha_k}|f|^2\,dx\,dt + s^3\lambda^4 \int_{\omega\times(0,T)} \xi_k^3|\psi_k|^2\,dx\,dt\right), \tag{11}$$

where $I(\psi_k,\xi_k,\alpha_k,Q_{\varepsilon,T})$ is given by (7) and

$$\begin{aligned} J(\psi_k,\xi_k,\alpha_k,\Sigma) = 2(-1)^{k+1}\Bigg(& s^3\lambda^3 \int_\Sigma \xi_k^3|\nabla\beta|^2\nabla\beta.\nu|\psi_k|^2\,d\sigma\,dt + (-1)^k 2s\lambda^2 \int_\Sigma \xi_k \frac{\partial\psi_k}{\partial\nu}|\nabla\beta|^2\psi_k\,d\sigma\,dt \\ & + 2s\lambda \int_\Sigma \xi_k(\nabla\beta.\nabla\psi_k)\frac{\partial\psi_k}{\partial\nu}\,d\sigma\,dt - s\lambda\int_\Sigma \xi_k|\nabla\psi_k|^2(\nabla\beta.\nu)\,d\sigma\,dt \\ & + (-1)^k \int_\Sigma \frac{\partial\psi_k}{\partial\nu}\partial_t\psi_k\,d\sigma\,dt - s^2\lambda\int_\Sigma \xi\alpha_t\nabla\beta.\nu|\psi|^2\,d\sigma\,dt\Bigg) \end{aligned}$$

and

$$J(\cdot,\cdot,\cdot,\Sigma_{\varepsilon,T}) = J\big(\cdot,\cdot,\cdot,\Sigma^D_{\varepsilon,T}\big) + J\big(\cdot,\cdot,\cdot,\Sigma^N_{\varepsilon,T}\big) + J\big(\cdot,\cdot,\cdot,C^1_{\varepsilon,T}\big) + J\big(\cdot,\cdot,\cdot,C^2_{\varepsilon,T}\big). \tag{12}$$

Treatment of boundary terms and passing to the limit in ε: Since $\beta = 0$ on Γ then $\alpha_0 = \alpha_1$, $\xi_0 = \xi_1$ and $\psi_0 = \psi_1$, we deduce that:

$$\psi_0 = \psi_1 = 0, \qquad \frac{\partial \psi_0}{\partial \nu} = \frac{\partial \psi_1}{\partial \nu} \quad \text{on } \Gamma_D^\varepsilon, \qquad \frac{\partial \psi_0}{\partial \nu} = -\frac{\partial \psi_1}{\partial \nu}, \qquad |\nabla \psi_0| = |\nabla \psi_1| \quad \text{on } \Gamma_N^\varepsilon,$$

which implies that:

$$\sum_{k=0}^{1} J\big(\psi_k, \xi_k, \alpha_k, \Sigma_{\varepsilon,T}^D\big) = \sum_{k=0}^{1} J\big(\psi_k, \xi_k, \alpha_k, \Sigma_{\varepsilon,T}^N\big) = 0. \tag{13}$$

On C_ε^l, $l = 1, 2$, we use the density of $D(-\Delta) \cap C^1(\overline{\Omega}) \oplus \text{span}\{r^{\frac{1}{2}} \sin \frac{\theta}{2}, r^{\frac{1}{2}} \cos \frac{\theta}{2}\}$ in $D(-\Delta)$, this allows us to write u in the form (2) with $u_R(\cdot, t) \in C^1(\overline{\Omega})$ for all $t \in (0, T)$, which implies that, for any $t \in (0, T)$:

$$\psi_k(\cdot, t) = O(\sqrt{\varepsilon}) \quad \text{and} \quad \big|\nabla \psi_k(\cdot, t)\big| = O\left(\frac{1}{\sqrt{\varepsilon}}\right).$$

Using the continuity of α_k and ξ_k, one can have:

$$\lim_{\varepsilon \to 0} \sum_{k=0}^{1} J\big(\psi_k, \xi_k, \alpha_k, C_\varepsilon^l\big) = 0, \quad l = 1, 2. \tag{14}$$

Then from (12)–(14), we deduce that:

$$\lim_{\varepsilon \to 0} \sum_{k=0}^{1} J(\psi_k, \xi_k, \alpha_k, \Sigma_{\varepsilon,T}) = 0.$$

Finally, by Lebesgue's theorem, we have:

$$\lim_{\varepsilon \to 0} \sum_{k=0}^{1} I(\psi_k, \xi_k, \alpha_k, Q_{\varepsilon,T}) = \sum_{k=0}^{1} I(\psi_k, \xi_k, \alpha_k, Q_T).$$

To achieve the proof of Theorem 2.1 we use, as in [4,6], the usual technics for Carleman estimates and the fact that

$$\xi_1 \leqslant \xi_0 \quad \text{and} \quad e^{-s\alpha_1} \leqslant e^{-s\alpha_0}.$$

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References

[1] A.H. Belghazi, F. Smadhi, N. Zaidi, O. Zair, Carleman inequalities for the two-dimensional heat equation in singular domains, Math. Control Rel. Fields 2 (4) (2012) 331–359.
[2] L. Bourgeois, A stability estimate for ill-posed elliptic Cauchy problems in a domain with corners, C. R. Acad. Sci. Paris, Ser. I 345 (2007) 385–390.
[3] P. Cornilleau, L. Robbiano, Carleman estimates for the Zaremba boundary condition and stabilization of stabilization of waves, arXiv:1110.5164v2, 2012, pp. 1–37.
[4] E. Fernández-Cara, M. González-Burgos, S. Guerrero, J.P. Puel, Null controllability of the heat equation with boundary Fourier conditions: The linear case, ESAIM Control Optim. Calc. Var. 12 (3) (2006) 442–465.
[5] Xiaoyu Fu, Logarithmic decay of hyperbolic equations with arbitrary small boundary damping, Commun. Part. Diff. Eq. 34 (7–9) (2009) 957–975.
[6] A. Fursikov, O.Yu. Imanuvilov, Controllability of Evolution Equations, Lecture Notes Ser., vol. 34, Seoul National University, Korea, 1996.
[7] P. Grisvard, Contrôlabilité exacte des solutions de l'équation des ondes en présence de singularités, J. Math. Pures Appl. 68 (2) (1989) 215–259.
[8] E. Shamir, Regularization of mixed second-order elliptic problems, Israel J. Math. 6 (1968) 150–168.